PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 67 FEBRUARY, 1941 No. 2

TECHNICAL PAPERS

AND

DISCUSSIONS

Published monthly, except July and August, at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York, N. Y. Reprints from this publication may be made on condition that the full title of Paper, name of Author, page reference, and date of publication by the Society, are given.

Entered as Second-Class Matter, September 23, 1937, at the Post Office at Lancaster, Pa., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 5, 1918.

Subscription (if entered before January 1) $8.00 per annum. Price $1.00 per copy.

CURRENT PAPERS AND DISCUSSIONS

		Discussion closes
Analysis of Legal Concepts of Subflow and Percolating Waters. *C. F. Tolman and Amy C. Stipp*	Dec., 1939	
Discussion	Feb., Apr., May, June, 1940	Closed*
Miniature System of First-Order Alinement and Triangulation Control. *Floyd W. Hough*	Dec., 1939	
Discussion	Apr., June, 1940, Feb., 1941	Closed
Pressure-Momentum Theory Applied to the Broad-Crested Weir. *H. A. Doeringsfeld and C. L. Barker*	Dec., 1939	
Discussion	Mar., Apr., June, 1940, Feb., 1941	Closed
Axioms in Roadway Soil Mechanics. *Henry C. Porter*	Feb., 1940	
Discussion	Sept., Oct., 1940	Closed*
Foundation Experiences, Tennessee Valley Authority: A Symposium	Mar., 1940	
Discussion	May, June, Sept., Dec., 1940, Jan., Feb., 1941	Closed
Design of Hinges and Articulations in Reinforced Concrete Structures. *George C. Ernst*	Apr., 1940	
Discussion	Oct., 1940, Feb., 1941	Closed
Permissible Composition and Concentration of Irrigation Water. *W. P. Kelley*	Apr., 1940	
Discussion	Sept., Nov., 1940, Feb., 1941	Closed*
Masonry Dams: A Symposium	May, 1940	
Discussion	Sept., Oct., Nov., Dec., 1940, Jan., Feb., 1941	Closed*
Theory of Elastic Stability Applied to Structural Design. *Leon S. Moisseiff and Frederick Lienhard*	Jan., 1940	
Discussion	Nov., 1940, Feb., 1941	Closed*
Model Tests, Bridge Pier Supported on Long Steel Piles. *Thomas F. Comber, Jr., and John M. Coan, Jr.*	June, 1940	
Discussion	Jan., 1941	Closed*
Concrete Specifications, Pt. 2	June, 1940	
Discussion	Nov., Dec., 1940, Feb., 1941	Closed
Maximum Probable Floods on Pennsylvania Streams. *Charles F. Ruff*	Sept., 1940	
Discussion	Dec., 1940, Jan., Feb., 1941	Mar., 1941
Superstructure of Theme Building of New York World's Fair. *Shortridge Hardesty and Alfred Hedefine*	Sept., 1940, Feb., 1941	Mar., 1941
The Grand Central Terminal in Perspective. *William J. Wilgus*	Oct., 1940	
Discussion	Dec., 1940, Jan., Feb., 1941	Mar., 1941
Earthquake Stresses in the San Francisco-Oakland Bay Bridge. *Norman C. Raab and Howard C. Wood*	Oct., 1940	
Discussion	Dec., 1940	Mar., 1941
Transatlantic Seaplane Base, Baltimore, Maryland. *W. Watters Pagon*	Oct., 1940	
Discussion	Feb., 1941	Mar., 1941
An Investigation of Steel Rigid Frames. *Inge Lyse and W. E. Black*	Nov., 1940	
Discussion	Jan., Feb., 1941	Apr., 1941
Reliability of Station-Year Rainfall-Frequency Determinations. *Katharine Clarke-Hafstad*	Nov., 1940	
Discussion	Jan., Feb., 1941	Apr., 1941
Cavitation in Outlet Conduits of High Dams. *Harold A. Thomas and Emil P. Schuleen*	Nov., 1940	Apr., 1941
Fort Peck Slide. *T. A. Middlebrooks*	Dec., 1940	May, 1941
Plastic Theory of Reinforced Concrete Design. *Charles S. Whitney*	Dec., 1940	
Discussion	Feb., 1941	May, 1941
Expansion of Concrete Through Reaction Between Cement and Aggregate. *Thomas E. Stanton*	Dec., 1940	
Discussion	Feb., 1941	May, 1941
Analysis of Statically Indeterminate Trussed Structures. *O. T. Voodhigula*	Jan., 1941	May, 1941
Laboratory Investigations of Soils at Flushing Meadow Park. *Donald M. Burmister*	Jan., 1941	May, 1941
Concrete in Sea Water; A Revised Viewpoint Needed. *Homer M. Hadley*	Jan., 1941	May, 1941
Dynamic Stress Analysis of Railway Bridges. *R. K. Bernhard*	Jan., 1941	May, 1941
Experiences in Operating a Chemical-Mechanical Sewage Treatment Plant. *George J. Schroepfer*	Jan., 1941	May, 1941
Rigid Frames Without Diagonals (The Vierendeel Truss). *Louis Baes*	Jan., 1941	May, 1941
Hydraulics of Sprinkling Systems for Irrigation. *J. C. Christiansen*	Jan., 1941	May, 1941

NOTE.—The closing dates herein published are final except when names of prospective discussers are registered for special extension of time.

* Publication of closing discussion pending.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

SOME ECONOMICS OF AIRPORTS

By W. Watters Pagon,[1] M. Am. Soc. C. E.

Synopsis

So much has been written regarding the location and the engineering design of airports that the writer has chosen rather to discuss some of the economic factors relating to the growth in size and in density of use of existing ports. This paper also deals largely with ports intended for transport planes. Until recently, the number of airports had reached and declined from a saturation point. Under the stimulus of war there is now (1941) a renewed growth, and a higher saturation value, although at present there is no reliable indicator to serve as a guide. For example, many state planning bodies are compiling lists of airports to be built progressively as the demand requires, and these will be correlated into a nation-wide plan; but the problem facing the present-day designer is primarily one of improvement and enlargement of present facilities.

Growth of Airplane Uses

The Entire Country.—Naturally the first statistical indicator is the growth in use of planes, and for determining this growth the designer must rely on the rates of growth of other means of transportation. Such means fall roughly into two classes: (*a*) The common carrier, and (*b*) the private conveyance. Indicators for these classes cannot be determined upon comparable bases, for lack of detailed information, but they will be set up from available information and utilized to the extent of their evident limitations.

In order to correlate such widely different sources of data one may adopt a Pearl biometric type of curve as indicative of growth of all means of transportation, as well as of biometric growth. Let:

f = numerical factor used to multiply u (having the dimension 1/time);

u = age in any units of time;

k = ratio of number of growth units to the (assumed) saturation number, with values between 0 and 1;

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by June 15, 1941.

[1] Cons. Engr., Baltimore, Md.

v, y, y' = number of growth units, measured to different base lines, as described below;

b = number of growth units at saturation; that is, at the time of maximum growth, which is (mathematically speaking) at infinite time;

$\log_e c$ = time distance between two origins, as described subsequently;

a = number of growth units (from a previous life) at inception of the growth function that is now operating.

If the origin is chosen at that year in which the number of growth units equals one half of the number at saturation, the result is a symmetrical S-shaped curve

$$f\,u = \log_e \left(\frac{k}{1 - k} \right) \qquad (1)$$

The abscissa, u, is measured plus from this origin with increasing years; ordinates, v, measured from the origin, are plus when u is plus, and ordinates, y', measured from a base line, are always plus; the factor, k, is the ratio of y' to the total number, b, at saturation. To another origin, at distance $\log_e c$ to the left of this midpoint origin the equation is

$$f\,x = \log_e c \left(\frac{k}{1 - k} \right) = \log_e c + \log_e \left(\frac{k}{1 - k} \right) \qquad (2)$$

The ordinate $y' = v + \frac{b}{2}$. If the curve begins tangent to a base line that is above the zero line by the distance, a, such that $y' = y - a$, Eq. 2 may be written

$$y = a + \frac{b}{1 + c\,e^{-fx}} \qquad (3a)$$

but in all that follows a equals zero, and $c = 1$; that is,

$$\frac{y'}{b} = k = \frac{1}{1 + e^{-fu}} \qquad (3b)$$

To use such a formula it is necesssry to know the saturation number, b, which of course it is impossible to know until all growth has ceased. However, if a smoothed curve through plotted data indicates a reversal of curvature, it may tentatively be assumed that the point of contraflexure is the aforementioned first origin; that is, $\frac{b}{2}$ above the base line. At the midheight point the slope of the tangent $\left(\text{that is, } \frac{dk}{dx}\right)$ is maximum, and is $\frac{1}{4} f$. In general, the slope is equal to $f\,k\,(1 - k)$, and therefore varies as a parabola referred to k. The annual increment, when $f = 0.3$, is $\frac{k_n\,(1 - k_n)}{k_n + 2.86}$, and the annual growth ratio is $\frac{k_{n+1}}{k_n} = \frac{1}{0.26\,k_n + 0.74}$, and therefore declines every year.

In Fig. 1 such a curve is shown, marked a, as well as others to be described. The vertical scale represents k; the horizontal scale represents $f\,u$ to the arbi-

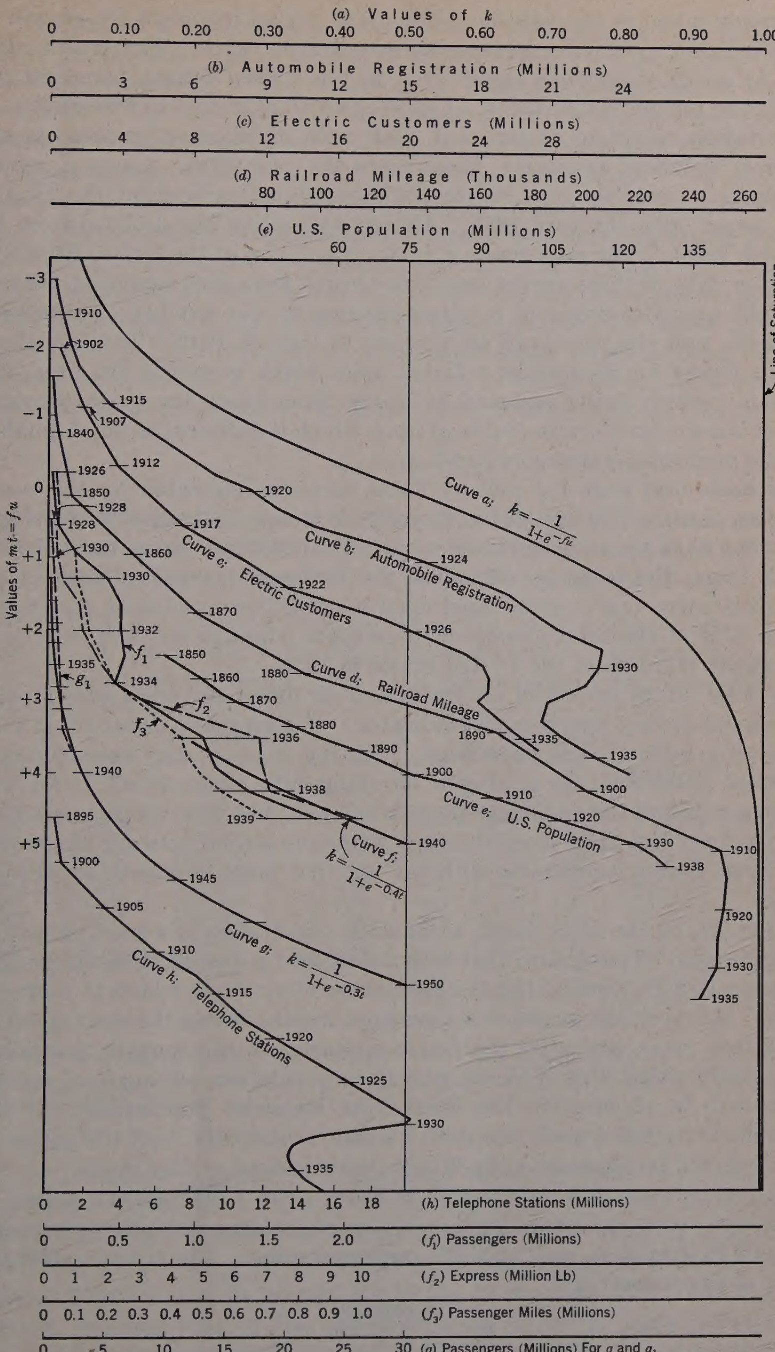


FIG. 1.—GROWTH INDICATORS (t EQUALS TIME IN YEARS, AND m IS A NUMERICAL FACTOR OF THE SAME TYPE AS f)

trary values noted at the bottom. The other curves represent the growth of various indicators, with abscissas as marked on each curve. The first of these (curve *b*) shows automobile registration in the United States, based on the year 1924 as the midheight point. Curve *c* (which is foreign to a discussion of transportation except in so far as it deals with distribution) represents the number of electrical customers, based upon the year 1926. Curve *d*, which probably is a complete one, represents the railroad mileage of the United States, based upon the year 1885. Already this curve has declined from its saturation point. The next curve, *e*, is the population of the United States.

To the right of these curves are three broken lines that represent, respectively, the annual number of airplane passengers, the number of passenger miles flown, and the pounds of air express, to include 1940. Through these curves is drawn an S-curve as a frame upon which to extend the data, the saturation number being assumed at about three times the present count. Beside it is a similar S-curve (curve *g*) upon which is redrawn, to much smaller scale, the number-of-passengers curve, g_1.

The horizontal scale for each of these curves is indicated by the years marked on them. The first curve (curve *f*) is merely indicative of the slope, because the data are so irregular as to permit almost any curve to be drawn through them, the irregular effects of the business depression being clear. The second curve (curve *g*) is based upon a saturation number of passengers equal to 12% of the total number of passengers (through and commutation) on the steam railroads of the United States in 1939.

The total future growth of air travel and air use is any one's guess. The data accumulated are clearly a poor indicator. If comparison is made first with the rate of growth of railroad mileage, then the engineer may count upon a continually increasing rate of growth for about fifty years, or say thirty-five years longer, before the midheight point is reached; but this comparison seems unfair, because in a little more than ten years the airline network has grown to a mileage that is comparable with one hundred years of growth of railroad mileage.

Seemingly, on the other hand, automobile registration is a good indicator, although it must be recognized that here the student is dealing with the private conveyance largely, whereas the commercial use of airplanes relates to common carriers. However, the automobile developed mostly during the short space of about fifteen years, and after the major expansion of the country had taken place (therefore at a time of slower growth in population, although, of course, some growth in registration has come from increased population). If the automobile indicator is used, one must assume, tentatively, that the period of half-growth will be approximately fifteen years, instead of fifty years.

The first airplane curve (curve *f*) is based upon a half-growth period of about fifteen years, and would yield as saturation values: 4,800,000 passengers, 24,000,000 lb. of express, and 1,600,000 passenger miles. The equation (for the number of passengers) is

$$y' = \frac{4{,}800{,}000}{1 + e^{-0.4t}} \qquad (4)$$

in which t is the time in years from 1940.

The second curve (curve g) is based upon a half-growth of about twenty-five years, and a saturation number of 12% of the railroad passengers—that is, 60,000,000 passengers. The equations for express and passenger miles would be related. For this curve the abscissas and ordinates have been altered, as shown, the ordinates being one twelfth, and the abscissas four fifths.

These curves (curves f and g, Fig. 1) cover a wide range. The one reflects individual desire for transportation to a great extent; and the other is based upon common carrier transportation, which is influenced by a long period of growth in population, and a paucity of population in the early years. Because of the irregularity of the data it is impossible to fit any curve that is not open to serious objection, but since some crude basis of prediction is needed, the writer offers curve g for discussion, as being more likely of attainment. If it be correct there will be a half-growth period of about twenty-five years. In further discussion, therefore, this latter curve is used. Its equation is

$$k = \frac{1}{1 + e^{-0.3t}} \quad \ldots\ldots (5)$$

in which t is the time in years, measured from 1950. The assumed saturation value for passengers is 60,000,000 per yr; and, for express, 180,000,000 lb per yr. From this curve, the growths in Table 1(b) are derived.

TABLE 1.—Growth Studies Applying to Air Transportation

(a) Annual Traffic at Newark, N. J.[a]				(b) Future Growth in the United States (Eq. 5)				
Year	Passenger fares	Freight, in Lb: Mail	Freight, in Lb: Express	Year	Value of k	Ratio[b] $\frac{k_{n+1}}{k_n}$	No. of passengers	Express, in lb
1929	4,000	322,000	0	1939	0.0355	1.34	2,130,000	6,390,000
1930	30,000	905,000	47,000	1940	0.0475	1.33	2,860,000	8,580,000
1931	90,000	2,061,000	64,000	1941	0.0630	1.32	3,780,000	11,340,000
1932	91,000	1,730,000	213,000	1942	0.0840	1.31	5,040,000	15,120,000
1933	117,000	1,603,000	457,000	1943	0.1090	1.30	6,540,000	19,620,000
1934	123,000	1,511,000	758,000	1944	0.1420	1.29	8,520,000	25,560,000
1935	205,000	2,744,000	1,151,000	1945	0.1825	1.27	10,950,000	32,850,000
1936	265,000	3,800,000	2,175,000	1946	0.2320	1.25	13,920,000	41,760,000
1937	299,350	4,766,719	2,488,580	1947	0.2890	1.23	17,340,000	52,020,000
1938	355,123	5,259,013	2,783,667	1948	0.3550	1.20	21,300,000	63,900,000
....	......	....	....	1949	0.4255	1.18	25,530,000	76,590,000
....	......	....	....	1950	0.5000	1.15	30,000,000	90,000,000

[a] Prior to opening La Guardia Field. [b] Where $\frac{k_{n+1}}{k_n} = \frac{1}{k_n(1 - e^{-f}) + e^{-f}} = \frac{1}{0.26\,k_n + 0.74}$, for $f = 0.3$. k_{n+1} = value of k for the $(n + 1)$th year, etc. For $f = 0.3$ this ratio gives 35% annual growth at the start, and 0% at the end of the life, and for $f = 0.4$, 50% and 0% respectively.

The Individual Airport.—The foregoing values apply to the United States as a whole. There are no means for predicting the volume of traffic for each individual airport, because competition between nearby airports, local business conditions, etc., make a prediction far more hazardous. For the sake of discussion, however, the writer will apply the 1950 ratio—14 : 1—to individual airports, meaning that by the end of a period of eleven years the volume of traffic will be fourteen times as great as in 1939. After five years Baltimore (Md.) exceeds such a curve by about 15%.

As indicative of such growth the traffic per annum at Newark, N. J., is given in Table 1(*a*). Transfer of the airlines to La Guardia Field caused an abrupt drop after 1938.

Density of Traffic

There is little experience to guide a designer in determining how many planes can be landed and discharged from an airport.[2] Such experience as there is points to a minimum time between take-offs of about $2\frac{1}{2}$ min. This would indicate a rate of 24 planes per hr, if systematically scheduled. Probably if double-width runways are provided this time can be reduced materially —perhaps by one half; but this is questionable.

If transport planes in the next ten years average 20 passengers each, landing or taking off from terminal airports, one may consider the probable passenger load to be roughly 500 per rush hr. For intermediate airports one can assume a capacity of fourteen times the values for 1939 as being the growth up to about the year 1950.

Size of Airplanes

There is no basis for predicting the growth in the size of aircraft that is not subject to much criticism. The growth curve may quite likely be an **S**-curve such as has been analyzed already; but the constants which determine the curve are indeterminate for want of basic data. Perhaps one may draw some conclusions from the rate of growth in speed, as indicated by the Schneider Cup Races, but it should be noted that in many cases the winner was a seaplane. For all winning planes, whether land or seaplanes, the horsepower increased roughly along a biometric curve. Similarly the top speed, when plotted, approximates such a curve and shows clearly the flattening off of the curve as it approaches what (in the light of present knowledge of the compressibility of air) is probably the maximum speed attainable in flight.

Airplane dimensions are more difficult to plot because they depend so largely upon the individual ideas of the designers of the differing types manufactured. It is probable that the rate of growth in size of land planes is approaching a maximum, but it is also probable that seaplanes will continue to grow beyond the limits of the land planes. The restriction imposed by the size of available landing areas is already affecting the first class, whereas there are areas along the ocean coasts and the Great Lakes in which it is possible to take off and land a large seaplane within reasonable access to important business centers.

In 1936 the writer made the prediction for seaplanes given in Table 2. The Pan American Clipper ships operating across the North Atlantic, for instance, have the actual dimensions listed under 1939. However, airline operators consider these estimates to be too conservative.

One American seaplane manufacturer has constructed his factory with a 300-ft clear span, and 40-ft clear height, anticipating the need for this added space in the not distant future. A proposed German design has a wing span of

---

[2] Some pertinent information will be found in "Forschungsvergebnisse des verkerswissenschaftlichen Instituts für Luftfahrt an der Technischen Hochschule Stuttgart," by Carl Pirath, Verlag von R. Oldenbourg, Berlin, 1930.

289 ft, a carrying capacity of 170 passengers and crew, length 158 ft, gross weight 250,000 lb. Mr. I. Sikorsky[3] has stated his belief in a ship of 1,000,000-lb gross weight as an engineering possibility. Such a plane might have a wing span of 450 ft. An English writer envisages a 500-passenger seaplane of 400-ft wing span, which will cross the North Atlantic in 12 hr.

TABLE 2.—Predicted Growth in Seaplanes

Year	Span of wings, in ft	Over-all length, in ft	Height, in ft	Gross weight, in lb	Weight per sq ft, in lb	Tail span, in ft	Passengers (sleeping)
1935	130	90	25	52,000	23.5	40	....
1939[a]	152	106	27.57	82,500	28.7	45.77	40
1945	175	110	32	100,000	31	42	50
1950	200	130	34	140,000	35	43	75
1955	250	160	37	250,000	40	45	100

[a] Actual values for Pan American Clipper ships.

Size of Airports

Mathematical Analysis.—In this paper attention is centered on landing fields for landplanes. The size of an airport depends, first, upon the obstructions in the vicinity. Until such obstructions are removed the landing area is reduced by some multiple of the height of the obstruction. In 1925 it was officially ruled that no area could be used for take-off or landing purposes whose distance from an obstruction was less than seven times the height of the obstruction. This rule was supplanted by a ratio of 10 : 1 very shortly; then by 15 : 1; and at present by 20 : 1; but the new (tentative) obstruction ratios are 30 : 1 on runways not to be used for instrument landings,[4] and 40 : 1 for those on which instrument landings are proposed. In this paper, therefore, it will be assumed that all such obstructions have been removed, and only the net dimensions will be considered available.

The length of a runway is determined primarily by two factors: (*a*) The take-off (or landing) run, while the plane is still on the ground or is just hovering; and (*b*) the length required for a path that will just clear an officially defined obstacle. In general, with brakes and wing flaps, the take-off run exceeds the landing run, and the latter need not be considered. The actual ground run is very much less than the total length of runway.

Although all pilots are required to warm up their engines and test them before taking off, there have been numerous cases in which the engines failed before any considerable height was attained, and cases in which the plane was so heavily loaded that it was unable to take off at all. In the single-engine plane the pilot must do the best that he can if the engine fails, and select a spot on which to land; in the multi-engine plane some power is still available to lengthen the landing glide, and therefore to multiply the chances for a safe re-landing. In any event, the pilot must choose between a straight glide, in the same direction as the take-off, or an attempt to turn in the hope of

[3] *Journal of The Aeronautical Sciences*, December, 1936, Vol. 4, No. 2, p. 75.

[4] Airport Approach Standards (Tentative), *Civil Aeronautics Administration*, August 26, 1940.

again regaining the airport. In the turn he may also suffer, or gain, from side slip, which in effect increases the gliding angle.

No one can predict at what time the engine, or engines, will fail; hence the following analysis is predicated upon determining the size of field required such that a pilot may re-land by effecting a turn (of 180° or less), regardless of the point at which failure occurs.[5] Side slip is assumed to be zero. The take-off angle, the glide angle, the speeds, and other factors in relation to size and power, are related to certain fundamental characteristics of airplanes. The relations are not identical for all makes of plane; and hence the resulting answer is more a generalized than a specific one.

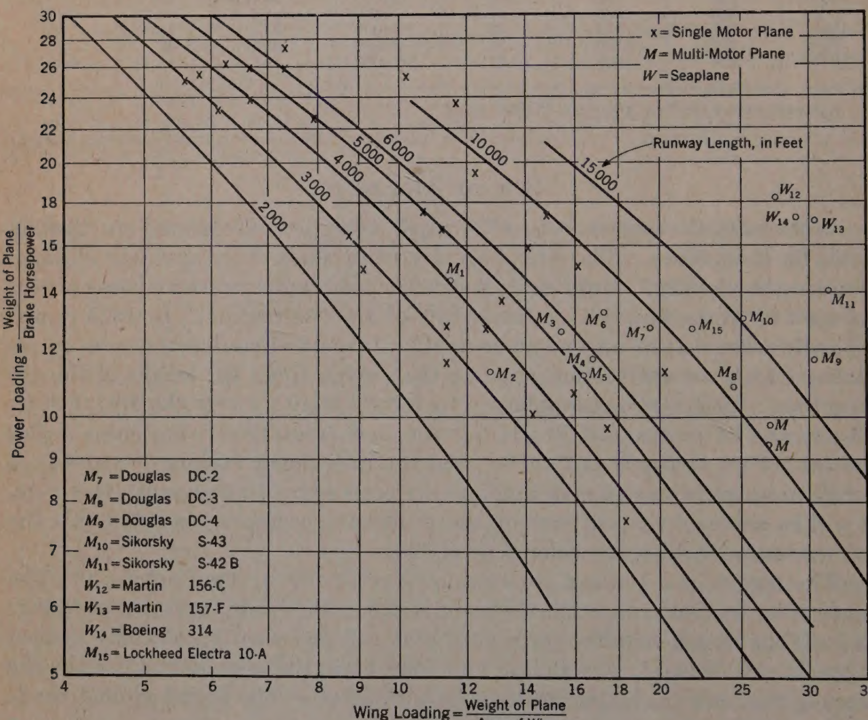


FIG. 2.—THEORETICAL LENGTH OF RUNWAYS

This answer appears in Fig. 2. The abscissas are the wing loading, and the ordinates the power loading. Wing loading is merely the weight of the plane per square foot of wing area; the power loading is the weight of the plane divided by the brake horsepower; both are factors that are known for various models, and are annually published in aviation magazines. In Fig. 2 the curves represent the length of runway on a square landing area within which a plane may re-land, with curving glide, based upon the two factors described. On this figure are also shown a number of present models, plotted according to their

[5] "On the Necessary Size of Aerodromes in Order that a Landing May Be Made if the Engine Fails When Getting Off," by H. Glavert, *British Reports & Memoranda No. 996*, January, 1926.

wing and power loadings. The points are classified as: Single engine planes, denoted by the letter, x; multi-engine planes, denoted by the letter, M; and seaplanes, denoted by the letter, W. Certain of these planes are identified by manufacturer's name and the model number; others can be identified by reference to published lists in aviation magazines.

It will be noted, first, that for fields larger than 5,000 ft the spacing of the curves becomes very small, so that there is little gain by enlarging the field beyond 5,000 ft. A second point to be noted is that there is relatively little gain from a 4,000-ft field to a 5,000-ft one. It is also evident, third, that none of the multi-engined planes can return safely if all engines fail, but that with one half, two thirds, three quarters, or more of the engine power available, they can return. To avoid confusion the only case shown is that for 100% loss of engine power.

The following conditions are precedent to the curves of Fig. 2: The field is level; the wind is zero; the ground run is assumed to be at the attitude of top speed (which is conservative); the coefficient of friction is 0.05; the climbing speed is 119% of the stalling speed (the latter, expressed in miles per hour, is 18.1 times the square root of the wing loading); and the glide angle is 5 : 1. These conditions are all reasonable.

Statistical Study.—Some years ago the writer compiled data on more than 800 existing airports. These data form the basis for Fig. 3, which represents the percentage of airports whose size is equal to, or less than, a given dimension or area. For instance, about two thirds of the group had runways between 2,000 and 2,500 ft in length; about seven eighths between 2,500 and 3,000 ft; and 98.5% less than 5,000 ft. A 2,500-ft runway at sea level is equivalent to

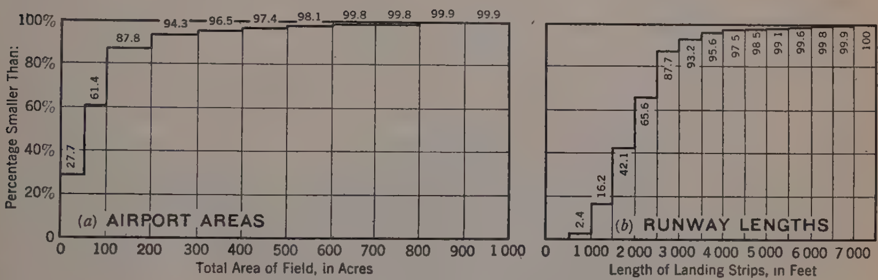


FIG. 3.—DESIGN CURVES FOR AIRPORT PLANNING

about 3,000 ft at El. 3,000. In Fig. 3, fields are included regardless of their elevation above sea level.

Probable Maximum Size.—In view of the large investment represented in airports it is scarcely likely that the size can be increased beyond 5,000 ft for even important ports, and it is certainly true that for many the size will not be enlarged beyond present dimensions because they cannot be enlarged.

In view of the utilization of multi-engined planes, the prevalence of wing flaps, and the possibility of other improvements, it seems that a fair conclusion to be drawn is that 5,000 ft probably will be the maximum size required in the future for airports intended for airplanes. Plane designers will be forced to

design planes that can use fields of such size safely. For seaplanes, however, two to four miles is the size under consideration.

For lighter-than-air ships, fields on which a hangar is to be provided will also require approximately 5,000-ft dimensions. For a hangar 1,000 ft long, with a 2,000-ft approach at either end, 5,000 ft will be required on the axis of the hangar. If, now, the hangar is rotatable, a 5,000-ft diameter will suffice. If means are developed for handling a ship into, and out of, a fixed hangar regardless of the velocity of cross winds, then the field may be restricted to 5,000 ft on the hangar axis, and perhaps 3,000 ft at 90° to the axis. Of course on such a field other space is needed for auxiliary buildings and storage.

The combined conclusion, then, is that probably a 5,000-ft square field, free of obstructions, at sea level (or of larger size for high altitudes) will be the maximum size required for major airports.

Density of Traffic.—Such a field will handle traffic up to the limit of perhaps 24 planes per hr on a single-width runway. Should double-width runways prove to be successful in operation the capacity will be increased. When congestion becomes too great a new field must be developed.

Field Arrangement

The question of field arrangement largely falls outside of the purview of this paper. The size of the airport, however, is determined in part by the presence of buildings upon it. The buildings should be located with reference to the runways so as to allow not less than 1,400 ft of clear width, centered on each runway, and if space permits, this width should be increased to avoid the presence of disturbing eddies whose intermittent character can create a severe hazard to planes landing or taking off. Such eddies may persist for a distance equal to 10, 20, or more times the height of the building, and they slowly widen as they move away from their source. Nearby hills, trees, and other outside obstructions are also a source of such eddies, and if the eddies are of the rolling type, with their axis horizontal, as results, for instance, from a grove of trees, their effect is to cause a sudden and rapid deviation in the direction of the wind near the ground surface which may cause the plane to appear as though forced down, or suddenly lifted. A field located adjacent to a body of water is also subject to variations in the rate of ascent of the rising, heated air, giving the same effect as a rolling eddy. The size of the field, therefore, should be determined with such eddies, and vertical currents, in mind.[6]

Airport Facilities

Administration.—The building that houses all means for serving the public will be called, for convenience, the "air station." Such a building will contain the airport administration, the airline ticket and accounting offices, the personnel for flight control, the personnel for meteorological service, the postal, express and such services, and finally (where required), customs, public health, and immigration services. Each of these will be considered in the light of present knowledge.

[6] "Wind Velocity in Relation to Height Above Ground," by W. Watters Pagon, *Engineering News-Record*, May 23, 1935.

The airport administration offices will probably be restricted to a space not greatly exceeding present requirements. Each airline will require not less than 12 ft of counter length and 200 sq ft of space behind it for the sale of tickets. If the airport is an intermediate one, an additional 600 sq ft of space will be required for accounting work, for teletypewriters, etc. If it is a terminal port, and, of course, if the main offices of the airline are at such a terminal airport, a large area may be required, or even an entirely separate building.

The size of the airport control room is a debatable matter. Opinions range from 144 sq ft to 256 sq ft, or more. An additional space, about equal in size, is needed for housing the equipment and accessories that need not be within the control room itself.

The space required for airway flight control and for meteorology increases from year to year. Only in recent years has there been need for control of traffic on the airways, but each year there will be increasing density of traffic, and increasing space in the station will be required for control. At present there is evidence that about 5,000 sq ft should be available, on one floor, close below the control room, for these facilities. In addition, "meteo" will require outside space, on the ground or on the roof, for instrument shelters. In Europe air police must also be accommodated.

The space required for postal and express services is dependent largely upon the type of operation as well as upon the volume of traffic. Where truck deliveries can be made, profitably, directly to and from a plane, little or no space is required in the station. With increasing volume the necessity will arise for inter-transit storage of express (between planes, or between planes and trucks); and in course of time the postal authorities will probably begin to demand space. These areas are completely indeterminate. The writer has attempted to obtain some clue from the express companies as to their requirements, but without success. Hence, the designer of an air station must rely entirely upon his imagination, and—having done so—should locate postal and express space so as to permit of a great expansion.

The space required for customs, immigration, and public health services will be considered only where there is likelihood of international airplane traffic. The writer has provided an area of 1,350 sq ft for these combined services at the Transatlantic Seaplane Base of the Baltimore (Md.) Municipal Airport[7] which, for the present, at least, has been satisfactory. The counter for customs inspection occupies 60 ft. When seaplanes that will carry more than 100 passengers are introduced, no doubt there will be need for additional space.

The spaces that occupy the greater part of the air station are those that serve the public generally, rather than the air traveler. Just as in the early days of the railroad, the general public has an intense interest in watching the operations of an airport. On Sundays and week ends, on holidays, on race days or days of exhibition flying or when some distinguished visitor may be seen, extraordinary crowds mill about the airport and into the station. Space for shelter, sanitary facilities, restaurants, etc., must be provided within reasonable limits.

---

[7] *Proceedings*, Am. Soc. C. E., October, 1940, p. 1488.

At present no one knows what are reasonable limits. During the week there are almost no visitors—only the traveling public; on week ends come the crowds. Table 3 gives some estimates of the number of persons who visit airports in different cities of the United States. In Continental Europe and in England the number is comparable and even larger. Many of the estimates are pure guesses, but they are still indicative.

TABLE 3.—THE NUMBER OF PERSONS VISITING AIRPORTS[a]

City[b]	An ordinary summer day	A summer week end	Largest recorded in one day
Akron, Ohio	1,500[c]	....	45,000
Washington, D. C. (Hoover)	500 to 600	1,000 to 1,200	1,500 to 2,000
Atlanta, Ga.	500	6,000[c]	50,000
Boston, Mass.	500 to 1,000	....	10,000
Buffalo, N. Y.	400 to 500	15,000[c]	75,000
Camden, N. J.	500	2,000 to 5,000	25,000
Chicago, Ill.	100	1,000[c]	200,000
Cincinnati, Ohio	200	5,000[c]	35,000
Cleveland, Ohio	3,000 to 4,000	28,000[c]	140,000
Columbus, Ohio	2,000 to 5,000	10,000 to 15,000	50,000
Dallas, Tex. (Love Field)	1,500 to 2,500	15,000 to 25,000	80,000
Dayton, Ohio	300	5,000	15,000
Des Moines, Iowa	100	2,000	30,000
Detroit, Mich.	....	10,000	75,000
Garden City, L. I., N. Y.	1,000	10,000	150,000
Grosse Ile, Mich.	....	5,000[c]	12,000
Hartford, Conn.	....	5,000[c]	75,000
Indianapolis, Ind. (Hoosier)	100	10,000 to 15,000[c]	75,000
Jacksonville, Fla.	100 to 500	10,000[c]	30,000 to 40,000
Kansas City, Kans.	5,000	10,000 to 30,000	259,221[d]
Kansas City, Mo.	....	5,000	20,000
LeRoy, N. Y.	100	20,000	86,000
Memphis, Tenn.	500	30,000	80,000
Miami, Fla. (Pan-American)	....	....	30,000
Minneapolis, Minn.	400 to 500	10,000	75,000
Newark, N. J.	2,500 to 3,500	5,000 to 10,000	60,000
Philadelphia, Pa.	200	1,000 to 10,000	10,000
Pontiac, Mich.	100	500 to 1,000[c]	50,000
St. Paul, Minn.	200	3,000	75,000
Syracuse, N. Y.	200 to 500	5,000	30,000
Terre Haute, Ind.	500	5,000	25,000
Toledo, Ohio	....	20,000	40,000
Utica, N. Y.	500	3,000 to 5,000	27,000
Wichita, Kans.	500	10,000	20,000

[a] "Airports," by Henry V. Hubbard, Miller McClintock, and Frank B. Williams, Harvard Univ. Press, 1930. [b] Special name of airport in parentheses. [c] Sunday only. [d] Estimate made on basis of 3 persons per automobile.

In his search for an indicator of the required extent of these public facilities the writer turned to railroad experience. Some years ago the American Railway Engineering Association collated information as to railroad stations throughout the United States, and derived diagrams[8] based upon the number of "rush-hour passengers." Of course, the rush hour is congested largely by commuters, for whom the station serves mostly as a gateway from train to street, and for whom little waiting space and few sanitary or restaurant facilities are required. Of course, also, the railroad station must be of a size to accommodate crowds assembled to greet friends or notable persons, without undue congestion.

It is probable that for ten years to come the crowds at an airport will far exceed those at a railroad station, in relation to the number of passengers; it is certain that for ten years to come there will be little business of commuting by airplane. As a result, the indicators described herein are all dependent upon a crude estimate, or even guess, as to the basic factor underlying them all;

[8] *Bulletin*, A.R.E.A., Vols. 24, 25, 26, 1923–1925.

but they possess this important characteristic: They are all consistent among themselves. If, then, the assumed basic factor proves to be correct, the station will be consistent in design; and in order to allow for error, the basic assumption should be underestimated. Then the design can be fortified by such planning that the facilities may be enlarged pro rata as the need arises.

From the railroad data the writer derived formulas for each of the public services. All depend upon the estimated number of rush-hour passengers (designated as p); some being in linear relation, others as the square root. Each curve may have a fixed, or initial, value as follows:

Concourse, in square feet	4,000 +	$4.4\,p$
Terrace, or outside area, in square feet		$300\,\sqrt{p}$
Men's smoking room, in square feet	500 +	$0.3\,p$
Women's rest room, in square feet	500 +	$0.6\,p$
Total inside seats, number		$17\,\sqrt{p}$
Men's toilets, in square feet	500 +	$0.5\,p$
Water closets, number		$0.55\,\sqrt{p}$
Urinals, number		$0.35\,\sqrt{p}$
Lavatories, number	4 +	$0.0044\,p$
Women's toilets, in square feet	400 +	$0.33\,p$
Water closets, number		$0.40\,\sqrt{p}$
Lavatories, number	6 +	$0.004\,p$
Telephone booths, number	1 +	$0.006\,p$
Telegraph office, in square feet	60 +	$3.5\,\sqrt{p}$
Dining room and restaurant, in square feet	500 +	$2\,p$
Seats, number	15 +	$0.08\,p$
Kitchen, in square feet	200 +	$1.2\,p$
News stands, in square feet		$11.5\,\sqrt{p}$

On the Continent it is customary to provide hotel facilities for the traveler, but this is not usual in the United States. However, some space should be provided, with appropriate toilet facilities, for visiting pilots, male and female, and there should be sleeping quarters for airport operating and for "meteo" personnel, these being dependent entirely upon the local conditions at the airport. If there are already such accommodations of suitable quality in the near vicinity they may be omitted, of course, except perhaps for operating and "meteo" personnel.

As time goes by an urgent need will be found for garaging automobiles of the traveling public, and such a garage should provide moderate accommodations for the airport personnel. In addition, large open areas should be set aside for parking the cars of the visiting public, with adequate roadways designed to avoid the congestion on visiting days.

Hangars

The area and size of hangars are problems that are peculiar to each airport. For intermediate airports only one such area is needed, with hangar storage

space as required for the local planes that base there. For terminal airports there must be hangars of number and dimensions suited to the requirements of the airlines basing there, in addition to the small local planes.

For important intermediate and terminal fields there is probably need for hangar area of 50,000 to 75,000 sq ft; for smaller airports, proportionately less. In one hangar space should be provided for repair shops and storage areas, and the writer suggests areas (in square feet) approximately as follows:

Engine overhaul shop	3,600
Engine storage	1,200
Wood shop	1,500
Paint shop	1,200
Metal shop	2,000
Propeller shop	800
Fabric shop	900
Machine shop	700
Forge shop and heating plant	1,000
Radio shop	500
Instrument shop	500
Parts storage	4,700
Accessory shop	300
Shop offices	1,000
Toilets, locker room, etc	2,000
Field equipment storage	3,000
	24,900

The ceiling height of the hangar should be from 30 to 35 ft, and of the shops preferably 18 to 20 ft. If the shops are situated within the hangar the offices, locker and toilet rooms, etc., may be placed in a second story. Some of the first-story shops may require additional height, and should be placed so as to utilize a part of the second story area.

Fueling

It is unnecessary to attempt a formula for the capacity of the fueling system. If servicing can be secured readily from nearby sources, it may be entirely proper to fuel directly from tank trucks; otherwise there should be underground storage of sufficient volume, and pumps of sufficient capacity, to minimize or to avoid delays in fueling. The problem is local to each airport.

Summary

In setting down the statistical and approximately mathematical data of this paper the writer (to use a slang expression) is "putting his neck out." Some of the predictions are based upon reasonable factual assumptions, some upon comparison with railroad and other experience, and some upon airport experience and the judgment that derives from such experience. Predictions of

growth are naturally very hazardous, because the growth of the air transportation industry has not yet progressed far enough for the determination of the trends; but, of even more importance, predictions are difficult because the irregularity of the data (resulting from the effect of the business depression) is such as to prevent the fitting of any accurate growth curve. Such curves have much value, however, in that they have been found to express growth relationships for other variables with considerable success; but they always must be projected with due caution, because it is noteworthy that sudden changes in conditions postulate the ending of the previous growth curve, and the inception of a new one. This is illustrated well in the curve, h, of Fig. 1, showing the number of telephone stations in the United States, in which there first appears a change of the curve during the World War, and then the slump due to the depression. It is highly unsafe, therefore, to project any such curves more than ten years in advance, and the result even at the end of ten years may be seriously in error.

The writer offers the data, and the deductions therefrom, for the sole purpose of inviting constructive criticism in order that, by concerted action, the industry may derive some idea, even though approximate, of the rate of growth, and the resulting requirements that airports must meet in the near future. He expressly omits a discussion of the engineering features of design because so much has already been written on the subject, much of it now obsolete, to be sure.

It is hoped, therefore, that the profession will criticize, freely, all the statements made, and in so doing will supplement the data and the deductions to the same end that the writer has in mind.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

VALUE OF PUBLIC WORKS

By J. P. Hallihan,[1] M. Am. Soc. C. E.

Synopsis

It has long been the theory in the United States that cyclic depressions in industry, due to lack of a market for the goods produced, resulting in large-scale unemployment, could be alleviated materially by increased activity in the field of public works—in the construction of facilities requiring no market except the approval of the people. The opinion was also widely held that the transfer of the man power released by industry into the operations of public works would be a simple process performed with no great loss of time, and that the equivalent employment would sustain the purchasing power of the nation until industry returned to a normal basis.

The nine years 1931 to 1939 have furnished an opportunity to test these theories, and it may be useful to review the record to determine to what extent they have been supported in practice.

The Volume of Public Works

There are only two sources of public works—those incident to the development or protection of the national resources or the requirements of national defense, defined as federal works, and those required by states and lesser political subdivisions to meet the needs of their respective communities, defined as non-federal works. In the larger sense, the public interest is dominant in all these works, since any construction or employment adds to the national income, but the federal works can be initiated and financed only by federal action, whereas the non-federal works are wholly dependent on the willingness of the people directly benefited to assume the necessary financial obligations.

Financing of Public Works

When it became apparent, late in 1930, that extraordinary efforts must be undertaken to find employment for the millions of workers dropped from the industrial rolls, the potentialities of state and municipal public works were

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by June 15, 1941.

[1] Director, Municipal Eng. Section, WPA, Washington, D. C.

explored. A survey conducted at the request of the Hoover Committee on Employment, by the American Engineering Council, through Engineering Committees voluntarily serving in each state,[2] reported in April, 1932, that a great reservoir of state and municipal projects for basic improvements existed in each state, typified by an estimate of $490,000,000 in the state of Michigan on a 5-yr program, and that this general proportion of needs to population was characteristic of the entire forty-eight states. The Committee further reported that, although a considerable proportion of the estimated volume had been in contemplation for some time and was in line as to general plans for early construction, it could not be financed through community sources because of dwindling tax revenues and a reluctant bond market. A considerable stimulus in the form of federal aid would be required to bring it within the range of availability.

Federal Aid

Federal aid was authorized by the Emergency Relief and Construction Act of 1932, through the Reconstruction Finance Corporation (RFC), in the form of loans on self-liquidating projects. As that definition admitted only projects for water supply and sanitation covered by rates to users, vehicular bridges and tunnels covered by tolls, and public transportation facilities covered by fares, it was not utilized in sufficient measure to assist the situation materially and was terminated on June 26, 1933. The National Industrial Recovery Act (NIRA) of June 16, 1933, creating the Federal Emergency Administration of Public Works (PWA) and authorizing loans on useful projects at 5% interest to the extent of 70% of the cost and a federal grant for 30%, brought more of the basic projects of municipalities into the construction stage.

The continuing increase in unemployment and in the number of persons coming on the relief rolls made it apparent that a more liberal stimulus must be provided to overcome the inertia of municipalities in the planning and presentation of public works. In consequence, the grant was raised to 45%, and the interest rate on loans was reduced to 4%.

In the meantime, as an emergency measure, the operations of the Federal Emergency Relief Administration (FERA), which made grants to states for direct relief and cultivated relief work projects, were supplemented by the Civil Works Administration (CWA), which put men at work on any kind of project that would provide work and succeeded in employing more than 4,000,000 workers in the winter of 1933–1934.

As a measure to furnish more immediate employment for unskilled labor, the Emergency Relief Appropriation Act (ERA) of 1935 authorized the creation of the Works Progress Administration (WPA) to take over the work programs of CWA and FERA and to operate as a general contractor in furnishing relief labor and a part of the material on projects designed by state, county, and municipal sponsors, suitable for force-account administration.

Thus, although the necessity of immediate expansion of public works was voiced in 1930, and became imperative in 1933, it was not until 1935 that it began to operate effectively in the works program. In the meantime, heavy

[2] President's Emergency Committee for Employment, Government Printing Office, Washington, D. C. 1931.

expenditures for direct relief and work expedients started the national income up from a low of 40 billion dollars in 1932 to 42.3 billions in 1933, and 50.1 billions in 1934. Reflecting the influence of advance in public works, it rose to 55.2 billions in 1935, 63.5 billions in 1936, and 71.9 billions in 1937. A reduction in the national income to 64.0 billions in 1938 is generally attributed to a reduction of federal aid in the latter half of 1937.

The federal aid extended to construction in the years from 1933 to 1939 is shown in Table 1. This table includes sponsor's contributions, which were an

TABLE 1.—EXPENDITURE OF EMERGENCY FUNDS IN CONSTRUCTION, AND EMPLOYMENT CREATED

No.	Description	1933	1934	1935	1936	1937	1938	1939
	(*a*) EXPENDITURES, IN MILLIONS OF DOLLARS							
1	Total, from emergency funds	400	1,696	1,547	3,203	2,434	2,284	2,905
2	Non-federal projects	301	1,187	925	1,870	1,455	1,559	1,803
3	Federal projects	40	363	304	450	243	184	208
	Sponsor's Funds:							
4	Works Progress Administration[a]	....	....	17	227	334	343	336
	Public Works Administration—							
5	National Industrial Recovery Act[b]	9	36	126	347	104	30	} 494
6	Emergency Relief Administration Act	....	....	20	213	184	166	
7	Loans[b]	50	110	155	96	114	69	198
	(*b*) EMPLOYMENT, IN THOUSANDS OF PERSONS							
8	Non-federal employment	285	1,484	1,241	1,853	1,278	2,209	2,100
9	Federal employment	21	123	110	244	178	149	150
10	Total employment created	306	1,607	1,351	2,097	1,456	2,358	2,250
	(*c*) COST, IN DOLLARS PER PERSON EMPLOYED							
11	All agencies	1,307	1,055	1,145	1,527	1,672	969	1,291

[a] From "Summary Charts and Statistics," WPA. [b] Does not include railroad nor limited-dividend housing loans.

integral part of PWA and WPA projects, as compiled by the Construction Statistics Section, WPA. The PWA operated four months in 1933, CWA operated two months in 1933, and WPA operated five months in 1935 (see Table 1).

FINANCIAL LIMITATIONS OF MUNICIPALITIES

The delay in organizing a public works program was not due to any lack of useful projects. Many other considerations intervened between promise and performance, such as (1) the rapidly increasing poverty of municipalities, (2) statutory legislation, and (3) established procedure.

(*1*) *Rapidly Increasing Poverty of Municipalities.*—Cities derive their principal revenue from property taxes. In 1932 the assessed valuation of all real property in the United States was estimated at 163 billion dollars. In 1922, the corresponding value was 125 billion dollars and in 1912, 70 billion dollars (Bureau of the Census, 1932). In 1937 it was estimated by experienced ana-

lysts to have dropped to 90 billion dollars. In the business district of a typical large city the drop was 40%. This condition made it impossible for many cities to furnish the share of the cost required by the federal agencies, either from tax resources or by borrowing.

(2) *Statutory Legislation.*—In times of prosperity there is a tendency for communities to over-extend themselves in construction and to incur financial obligations beyond their ability to carry when tax revenues fall off. This tendency is controlled by statutory limitation of use of community credit and tax resources to a definite percentage of the current assessed valuation, generally about 10%. Exemptions from the bond limitations are made in the case of municipally owned public utilities that are made self-supporting by service charges to the users—as water, light, power, and public transportation facilities. Further restrictions on municipal borrowing, quite as effective as direct statutory and charter limitations, are found in state laws limiting investment of funds of savings banks, life insurance companies, trust companies, and sinking funds to bonds meeting the test of rigid financial standards. In many states municipal bond issues are automatically prohibited when tax delinquencies exceed a definite ratio of the current tax levy.

(3) *Established Procedure.*—Municipal improvement bonds must be submitted to the people at a general election and must be approved by at least a majority, and usually 60% of the total vote on the question. Even in times of prosperity, projects of recognized necessity may have to be submitted several times before approval is secured. In recent elections, in cities having more than 50,000 population, statistics secured by the Department of Commerce show that 65% of the bond issues were disapproved.

From the experience gained in this period of experimentation, it may be fairly stated that the theory that public works may be depended upon in times of depression to take up the slack in industrial employment is tenable only in part.

Effect on Construction Industry

The quick action necessary to make the theory effective practically is not possible of attainment under the foregoing conditions. Comparatively few of the projects presented to PWA in 1933 could be brought into line with respect to legal and financial requirements in time for approval under the authorization of funds for that program. Defects in bond issues required new legislative action in many states to make the bonds acceptable in the bond market. In this connection, the powers, limitations, and scope of action of the federal agencies conducting works programs of non-federal sponsorship should be understood.

Differences in Authority of Relief Agencies

Although both PWA and WPA are designed to stimulate the production of jobs for the unemployed, there is a considerable difference in their powers and limitations and, in consequence, in their scope of action.

The PWA may grant 45% of the cost of a public works project only when the sponsor has first furnished the remainder of the cost in the form of sequestered funds, bank credits, or definite commitment to purchase the securities to be

issued; but it may further assist the sponsor by direct purchase of the securities issued for his share which are stipulated to bear 4% interest with provision for amortization in thirty years or less. The work must be performed under contracts acceptable to PWA and under its engineering supervision, and must employ not less than 25% of relief labor. The direct authority of PWA thus extends down to the job.

The operations of WPA are on a different basis. It has no authority to make grants of money. It may meet the entire payroll on public works projects designed by the sponsor on which at least 95% of the federal funds expended for labor are applied to payment of labor certified from the relief rolls, and may also contribute toward the rental of specified equipment and purchase of materials in definite proportion to the volume of employment created. The sponsor designs and supervises the work and accepts full responsibility for completion in the event that the relief labor finds other work or WPA comes to the end of its appropriation. The WPA executes the work through its state administrators who have sole authority to select for operation, from a general list of projects approved by the President of the United States, those projects that are most suited to the current needs of the local relief program.

These differences in limitations and restrictions have resulted in a division of the public works assisted by these agencies along lines more or less determined by the quantity of material required by the project. Public buildings, schools, hospitals, water purification plants, sewage treatment plants, power plants, bridges, grade separations, and subways, which require a large proportion of manufactured material and skilled labor, averaging approximately 40% labor and 60% material, constitute the majority of PWA projects.

Because one of the determining rules of eligibility for WPA expenditures is that federal funds should be conserved for wages in the greatest degree consistent with proper execution of the job, the work undertaken runs largely to construction of roads, streets, dams, water mains, storm and sanitary sewers, airports, armories, landscaping of school grounds and parks, recreational facilities, drainage, and mosquito elimination, in which the labor-material proportion averages approximately 70% and 30%, respectively.

The works developed, however, were highly effective in retarding the disintegration of the construction industry, which was very seriously affected by the great drop in private construction. The highlights of the construction situation from 1925 to 1939 and the effect of construction employment on the national income are shown in Table 2.[3]

It may be observed from Table 2 that public works construction constitutes only about 20% of the total volume of construction in normal times (1925–1929), the remaining 80% being made up by private construction. In the depression period from 1930 to 1939, the total volume of construction fell to 53% of normal, but of this reduced volume 50% (2.3 times its normal proportion) was furnished by public construction. Taking the average expenditure in the 5-yr period from 1925–1929 as a base, the cumulative deficiency in

[3] "Construction Expenditures and Employment 1925–1936 and 1937 Compared with 1936," by Peter A. Stone, Construction Statistics Section, WPA. Corrected to December 31, 1939, in "National Income Produced," U. S. Dept. of Commerce.

private construction since 1929 is more than 53 billion dollars, whereas the cumulative excess in public construction is more than 4 billion dollars. Employment from private construction averaged only 41% of its former value, whereas public construction employment averaged 316% of normal.

TABLE 2.—CONSTRUCTION

No.	Description	Pre-Depression 1925	1926	1927	1928	1929
		(a) CONSTRUCTION EXPENDITURES,				
1	Total private construction	7,978	8,260	8,523	8,576	7,751
2	Total public construction	2,181	2,137	2,373	2,484	2,415
3	Grand total construction	10,159	10,397	10,896	11,060	10,166
	Comparison Between Normal and Depression Periods:					
4	All construction	Average, 10,536				
5	Private construction	Average, 8,218				
6	Public construction	Average, 2,318				
		(b) AVERAGE EMPLOYMENT, IN				
7	In construction, Annual	2,125	2,197	2,352	2,396	2,265
8	In construction, For the period	Average, 2,267				
9	In private construction, Annual	1,689	1,762	1,859	1,862	1,736
10	In private construction, For the period	Average, 1,782				
11	In public construction, Annual	436	435	493	534	529
12	In public construction, For the period	Average, 485				
		(c) NATIONAL INCOME,				
13	Totals[a]	72.7	74.9	73.7	77.6	81.1

[a] From "Construction Expenditures and Employment 1925–1936 and 1937 Compared with 1936," by 1939, in "National Income Produced"; U. S. Department of Commerce. National income for six years prior to for 1924, 67.9.

It will be noted in Table 2 that in the 5-yr period, 1930 to 1934, inclusive, there was a continuous drop in both public and private construction to a low in 1933; but at the end of 1933, public construction was still 80% of its normal volume, whereas private construction was less than 15%. The rise in volume thereafter was principally due to public construction which arrived, in 1939, to 176% of normal, whereas private construction is still only 45% of normal.

POSSIBILITIES OF PUBLIC WORKS

It appears, therefore, that, although the expansion in public works fell considerably short of expectations, in point of immediate availability, the results obtained in the maintenance of the construction industry fully demonstrate the high value of public works in the national economy.

It appears quite evident, also, that, in similar circumstances in the future, the immediate effect sought cannot be achieved unless there is present an ample reservoir of projects that have gone through the time-consuming preliminaries of engineering examination and legal and financial authorization,

and are ready to go into operation as soon as financial arrangements can be perfected. It is not easy to develop such a program in advance of need. The restrictions attending advance planning in municipalities and the development of construction programs are not likely to be materially reduced in the future.

xpenditures and Employment

Depression					Post-Depression				
1930	1931	1932	1933	1934	1935	1936	1937	1938	1939
Millions of Dollars									
5,379	3,422	1,411	1,175	1,363	1,982	3,005	3,589	3,268	3,687
2,726	2,512	1,878	1,827	2,619	2,579	3,261	3,034	3,373	4,091
8,105	5,934	3,289	3,002	3,982	4,561	6,266	6,623	6,641	7,778
Average, 4,862; average deficiency, 5,674; and total deficiency, 28,370.					Average, 6,374; average deficiency, 4,162; and total deficiency, 20,810.				
Average, 2,550; average deficiency, 5,668; and total deficiency, 28,340.					Average, 3,106; average deficiency, 5,112; and total deficiency, 25,560.				
Average, 2,312; average deficiency, 6; and total deficiency, 30.					Average, 3,268; average excess, 950; and total excess, 4,750.				
housands of Persons									
1,968	1,593	1,001	1,107	2,185	2,149	3,174	2,719	3,493	3,266
Average, 2,265; average deficiency, 2; and total deficiency, 20									
1,334	938	420	348	366	533	814	876	802	905
Average, 734									
634	655	581	759	1,819	1,616	2,360	1,843	2,691	2,361
Average, 1,531									
Billions of Dollars									
68.3	53.8	40.0	42.3	50.1	55.2	63.5	71.9	64.0	68.5

eter A. Stone, Construction Statistics Section, Works Progress Administration. Corrected to December 31, 925 was reported as follows: For 1919, 67.3; for 1920, 68.1; for 1921, 50.7; for 1922, 58.6; for 1923, 68.0; and

Accordingly, dependence must be placed on projects of national scope or local projects in which the national interest warrants federal participation in planning and financing.

There is no paucity of such projects. The nation has a definite interest in the elimination of 232,000 railroad grade crossings, most of which are in cities and are daily adding to the sum of fatalities and injuries from motor accidents. First-stage improvement on 2,000,000 miles of rural roads to make them passable in all weathers would affect, beneficially, the fortunes of 20,000,000 people and would be a sound investment of federal funds. It might be found justifiable, in view of their importance in national defense, to assist the railroads in catching up with their deferred maintenance of roadway and track, estimated by railroad authorities to require an investment of $400,000,000 annually on a 5-yr program.

Construction of wide thoroughfares through slum districts of large cities and the creation or betterment of public transportation facilities would merit federal assistance. The engineering surveys precedent to developing a program

for such projects should properly be under federal control and direction through the appropriate works agency. The consolidation, in 1939, of government construction agencies under a single control in the Federal Works Agency, an administrative betterment long urged by engineers, and a similar consolidation of government loaning agencies under the Federal Loan Agency, have made possible a continuous coordination of effort with the National Resources Planning Board in the preparation and financing of a program of federal public works designed to permit the orderly development of the nation's resources.

The machinery is available, therefore, to inventory and classify the public works of the nation and to determine in what degree the national benefit derived from basic non-federal projects justifies their inclusion in a federally controlled and federally aided program of such volume, variety, and territorial coverage as to be really serviceable in taking up the load of industrial unemployment.

Many of the most important civic projects are unable to proceed because funds are not available for the purchase or condemnation of the lands or right of way required, as with rehabilitation of blighted or slum areas, elimination of grade crossings, and widening of important traffic arteries. The powers of the loaning agencies might reasonably be adjusted to meet such emergencies by long-term, low-interest loans, or the federal government might acquire the lands directly and lease them to the communities on a long-term recapture basis.

Another handicap to development of a program of first-grade projects is the limitation of federal aid to an annual basis, leaving a community with no assurance that a project may be prosecuted continuously to completion as a usable asset. Such handicaps can readily be removed by suitable legislation, but while they exist they operate to prevent the development of a public works program of the first order of usefulness.

Conclusion

On a foundation of preparedness and with administrative machinery geared for prompt action and control, there is every reason to believe that public works can be brought much closer to expectation as a compensative employment element. To make a program of public works fully effective in assisting to take up the slack in industry, its full power must be thrown in promptly upon the inception of the depression. For that reason, federal works, which are immediately controllable upon congressional authorization, must be prepared to furnish the greater volume of projects.

Under the most favorable conditions, a certain time is required to bring non-federal projects into operation. In times of industrial depression, cities are always in a straitened financial position from loss of revenues and impaired banking credit. To cut down this time, arrangements for federal loans on a long-term, low-interest basis for programmed projects should be perfected in advance of need.

Acknowledgment

Acknowledgment is made to Peter A. Stone, chief of Construction Analysis Unit, WPA, for valued assistance in compilation of the construction and employment statistics used in this analysis.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

SIMPLIFIED THEORY OF THE SELF-ANCHORED SUSPENSION BRIDGE

By C. H. Gronquist,[1] Assoc. M. Am. Soc. C. E.

Synopsis

The theory of the self-anchored suspension bridge is presented in this paper in a simplified form that lends itself to a straight-forward and expeditious analysis of that type of structure.

The self-anchored suspension bridge, unlike the externally-anchored type, may be properly analyzed by the elastic theory, since the effect of distortions under live load is practically eliminated in this "closed" structure. The formulas which follow are therefore based on the elastic theory.

Vertical camber of the stiffening girder is effective in reducing cable and girder stress in the self-anchored bridge by the arch-like action of the cambered girder, and this factor is included throughout. Moreover, as the resultant stress in the girder, like that in an arch rib, is that produced by direct thrust and moment, influence lines for girder flange stress may be constructed as for moment about the kern points of the girder. This method of stress analysis is used in the computations, which illustrate the application of the theory of the self-anchored suspension bridge.

The formulas for the self-anchored suspension bridge have been derived for a three-span symmetrical bridge with continuous girders. By dropping the continuity terms, the formulas for girders hinged at the towers are obtained. It will be observed that these formulas, with girder camber and shortening eliminated, are the same as those for the externally-anchored suspension bridge analyzed by the elastic theory.

Constant girder moment of inertia within each span has been assumed; the effect of this approximation, as well as that of neglect of suspender elongation, is considered, as is the modification of the formulas for unsymmetrical and multiple spans. A method of correcting the value of cable and girder H as

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by June 15, 1941.

[1] Assoc. Engr., Robinson & Steinman, New York, N. Y.

computed by the elastic theory, in order to take into account the movement of the ends of the girder produced by its vertical deflection, is indicated.

Introduction

In the self-anchored suspension bridge the cable is attached to the stiffening girder at the outer ends of the side spans. Thus the horizontal pull of the cable is resisted by the girder in compression, and the anchorage foundations are subjected only to the upward vertical component of the cable stress, in addition to the girder reactions. This type of construction, consequently, permits the use of a suspension structure where foundation conditions do not justify the ordinary externally-anchored suspension bridge, with anchorages that must resist the horizontal pull of the cable.

There is some doubt as to whether the self-anchored suspension bridge was originated by an Austrian, Joseph Langer, in 1870, or whether that type of structure was covered by an American patent issued to Charles Bender in 1867. Although Bender's design and Langer's Wrsowic Bridge (1870) were of the self-anchored type, the cable was attached to the girder in the main span as well as at the outer end of the side spans of these bridges. The first truly self-anchored suspension bridge of the modern type was built at Lübeck, Germany, in 1899. The self-anchored suspension bridge has since been adopted extensively in Europe and in one instance in Japan. In the United States three almost identical examples of this type of bridge were built from 1926 to 1928 in Pittsburgh, Pa. Two bridges have also been built in the United States in which rolled beam girders and open-strand cables were used—the first in 1933 over the Little Niangua River in Missouri, and the second in 1939 with a 350-ft main span over the Wabash River in Indiana. In Guatemala a small self-anchored bridge was built in 1937 on the Pan-American Highway. (Howard Mullins, in 1936, supplied a list and bibliography of the self-anchored suspension bridges of the world.)[2]

Although several of these bridges are of moderately long span, ranging from 600 ft to 1,000 ft, most of them are approximately 400 ft, or less, in length of main span. The application of this type of structure to spans of less than 400 ft—in which, for light loading, rolled beams can be used as stiffening girders and open-strand construction for the cables, and where the shortness of span will lessen the disadvantage that deflection does not appreciably relieve girder moment—appears to offer distinct possibilities for improving appearance in bridges of this size in locations which would not be considered suitable for the externally-anchored suspension bridge.

So far as is known to the writer, no complete treatise on the theory of the self-anchored suspension bridge is available in English, although some valuable information has been published on the subject. (Computations for several self-anchored suspension bridge designs were presented by G. G. Krivoshein in 1930;[3] and, as stated, a review of the history and analysis of the characteristics of this type of structure was published by Mr. Mullins in 1936.[2]) The following

---

[2] *Engineering News-Record*, January 9, 1936, p. 45.
[3] "Simplified Calculation of Statically Indeterminate Bridges," by G. G. Krivoshein, Prague, 1930.

is offered as a simplified statement and extension of the existing treatments of the theory of the self-anchored suspension bridge in the belief that, with a more complete, simplified treatment available, the use of the self-anchored suspension bridge for spans of moderate length will increase.

Notation

The letter symbols in Fig. 1 and elsewhere in the paper are defined in the text where they first appear and, for convenience of reference, are listed alphabetically in the Appendix.

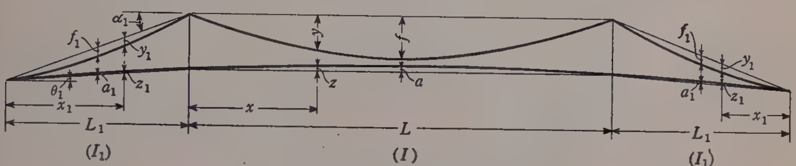


Fig. 1.—Notation Diagram

Dead Load

The stiffening girder, of symmetrical cross section and constant depth, of a self-anchored suspension bridge is subjected to no dead-load moment provided it is laid out during fabrication for dead-load position, and that the lengths of the towers, cable, suspenders, and girder are fabricated correctly to bring the girder exactly to this position after the application of the dead load. If the girder is cambered parabolically in any span, it will carry a portion of the uniformly distributed dead load as a parabolic arch, the total stress being axial. If the side-span cable is unloaded, the side-span girder must naturally be self-supporting for dead load, and when, in addition, the girder is continuous through the towers, it will be subjected also to dead-load bending stress in the main span. The latter type of structure is not considered herein; but by including this dead-load moment in the formulas that follow, they become applicable to that type of structure.

To determine the value of the horizontal component of cable and girder stress due to dead load, the expression for moment in the girder at the center of the main span may be written and placed equal to zero. In Fig. 2 let:

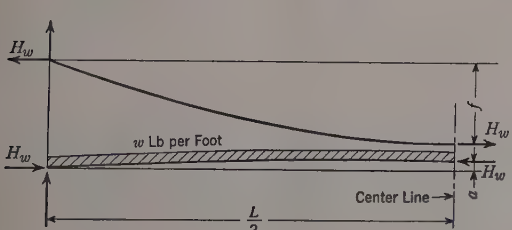


Fig. 2

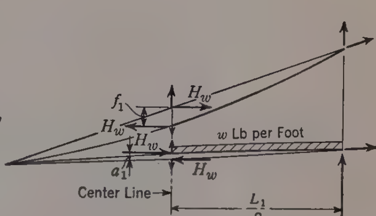


Fig. 3

H_w = horizontal tension in cable and compression in girder due to the dead load w; f = cable sag at the center of the main span; a = camber of the main-span stiffening girder; L = main-span length; and M_w = the bending moment

due to dead load. Then,

$$M_w = \frac{w\,L^2}{8} - H_w\,a - H_w\,f = 0 \quad\ldots\ldots(1a)$$

and

$$H_w = \frac{w\,L^2}{8\,(f + a)} \quad\ldots\ldots(1b)$$

The portion of the dead load carried by the girder as a parabolic arch (w_a) may be evaluated by taking the second derivative of the expression for the negative moment that tends to act on the girder as a result of its parabolic camber. The expression for this moment is:

$$M_w' = -H_w\,z = \frac{-4\,a}{L^2}(L\,x - x^2)\,H_w \quad\ldots\ldots(2a)$$

and

$$w_a = \frac{d^2}{dx^2}(-H_w\,z) = 8\,H_w\frac{a}{L^2} = \frac{w\,a}{f + a} \quad\ldots\ldots(2b)$$

in which: z = main-span stiffening girder ordinate; and x = abscissas along the center span, measured from the tower. The value of the side-span cable sag f_1 is obtained by writing, as in the main span, the expression for moment in the girder at the center of the span (see Fig. 3):

$$M_{w1} = \frac{w_1\,L_1^2}{8} - H_w\,(f_1 + a_1) = 0 \quad\ldots\ldots(3)$$

and therefore,

$$f_1 = \frac{w_1\,L_1^2}{w\,L^2}(f + a) - a_1 \quad\ldots\ldots(4a)$$

Again (see Eq. 2*b*):

$$w_{a1} = \frac{w_1\,a_1}{f_1 + a_1} \quad\ldots\ldots(4b)$$

Live Load

The application of live load to the bridge produces deflections of the cables and stiffening girders that change the dimensions of the structure. The changes in cable span lengths due to tower-top movement are negligible since they are small in proportion to the total span lengths. That the proportionately large changes in cable sag and girder camber have no effect on the moments in the girder may be seen by writing the general expression for moment in the girders (see Fig. 4):

$$M = T + M_D + M_0 - (H_w + H)\,(y + \eta) - (H_w + H)\,(z - \eta) \quad\ldots(5)$$

in which: M = the total resultant bending moment at any section x of the stiffening girder; T = bending moment at any section x due to continuity; $M_D = \frac{w\,L^2}{8}$ = simple-beam bending moment for dead load; M_0 = simple-beam bending moment due to a live load p; H = horizontal tension in the cable and compression in the girder due to a live load p; y = main-span cable ordinate; and η = deflection of the stiffening girder at any section x. It will be observed that η cancels in Eq. 5. Since the dead-load moment equals 0,

$$M_D - H_w\,(y + z) = 0 \quad\ldots\ldots(6a)$$

and, in any span,

$$M = T + M_0 - H\,(y + z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6b)$$

The load taken by the girder in moment is equal to the negative of the second derivative of that moment.

$$p_g = \frac{-d^2 M}{dx^2} = p - \frac{8\,(f + a)\,H}{L^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

This is the difference between the applied distributed load p and that taken by the cable and by the girder as an "arch."

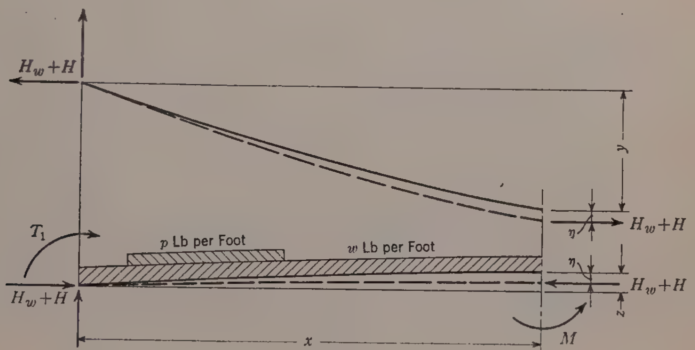


FIG. 4

Thus it may be stated that the self-anchored suspension bridge is an independent, closed structure, in its action under live load, in many ways more nearly resembling an inversion of the tied arch than an externally-anchored suspension bridge. Whereas the stresses in the latter type of structure are greatly affected by the changes in the cable sag produced by deflection, those of the self-anchored bridge are not so affected due to the simultaneous change in girder moment of H multiplied by the deflected z. Because the sum of $(y + z)$ or $(f + a)$ in any span is constant during deflection, neglecting suspender elongation, the value of the dead load H_w remains unchanged during deflection; the dead load carried by the girder-arch is decreased, that by the cable is increased.

Inasmuch, then, as the deflection of the bridge has no appreciable influence over the dead load H_w or the live-load moment in the girders, except over the latter in the side span when the side-span cable is unloaded, the self-anchored suspension bridge may be analyzed by application of the elastic theory. The following assumptions have been made:

(1) The cable curve before deflection is a parabola whose equation, with the closing chord as the x-axis and the origin at one end, is

$$y = \frac{4\,f\,x}{L^2}\,(L - x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

in all spans.

(2) The girder camber is parabolic. (For a girder that lies in a straight line the a of the formulas (Eqs. 1 to 7) is made equal to 0.) With the closing

chord as the x-axis and the origin at one end:

$$z = \frac{4\,a\,x}{L^2}(L - x) \qquad (9)$$

in all spans.

(3) The average moment of inertia of the stiffening girder throughout each span is used as the constant I of that span.

(4) The work of deflecting the tower top is negligible.

The moment of continuity at any point may be written:

$$T = T_1'\left(\frac{L - x}{L}\right) + T_2'\frac{x}{L} + H\,(f + a)_m\left[e_1\left(\frac{L - x}{L}\right) + e_2\frac{x}{L}\right] \qquad (10)$$

in which the T'-terms represent the effect of continuity on the independent girder (cable removed), and the e-term represents the continuity effect on the girder of the assumed uniformly distributed load of $\frac{8\,(f + a)\,H}{L^2}$ from the sum of the suspender forces and the equivalent of the effect of the negative moment, $-H\,z$, induced in the girder as an arch. Expressions for the values of the T'-terms are readily available in standard texts or may be determined from the Theorem of Three Moments. From that theorem the expression for e for a three-span symmetrical bridge, continuous through the main towers but hinged at the outer ends of the side spans, may be determined as

$$e = e_1 = e_2 = \frac{2\,(1 + i\,r\,v')}{3 + 2\,i\,r} \qquad (11)$$

at the main towers and throughout the main span; in the side span the e-term varies from 0 at the free end to $e_{1,2}$ at the tower. In Eq. 11 (see Appendix): i = the ratio of moments of inertia, $\frac{I}{I_1}$, main span to side span; r = ratio of span lengths $\frac{L_1}{L}$, side span to main span; and v' = ratio of total sag and camber $\frac{f_1 + a_1}{f + a}$, side span to main span.

The general moment equation applying to all spans may be written in the following form as a working equation:

$$M = (T' + M_0) + H\,[e'\,(f + a)_m - (y + z)] \qquad (12)$$

In this equation e is written with a prime to indicate that it varies as a straight-line function within any span, and $(f + a)$ is written with the subscript m to indicate that it takes the main-span value in all spans; T', similarly to e, varies as a straight-line function within any span.

The vertical shear in the girder corresponding to this moment is in general form for all spans:

$$V = \frac{dM}{dx} = \left[\frac{(T_2' - T_1')}{L} + V_0\right] - H\left[4\left(\frac{f + a}{L^2}\right)(L - 2\,x) - (f + a)_m\frac{(e_2 - e_1)}{L}\right] \qquad (13)$$

in which V_0 = simple-beam shear due to live load.

Determination of H for Live Load

To derive the expression for H by the method of the redundant member, the cable may be conceived as cut, with the horizontal component of cable stress H the redundant. Then:

$$H = -\frac{\sum \frac{S' \, u \, L}{A \, E} + \sum \int_0^L M' \, m \frac{dx}{E \, I}}{\sum \frac{u^2 \, L}{A \, E} + \sum \int_0^L m^2 \frac{dx}{E \, I}} = \frac{N}{D} \quad \text{(14)}$$

In Eq. 14 S', the direct stress in the various members of the system with the cable cut, equals zero throughout; M', the moment in the members of the system with the cable cut, is the moment in the independent girder in each span; $M' = T' + M_0$ in all spans. The term u denotes the direct stress in the members of the system corresponding to a horizontal component of stress in the cable and girder of unity. In the cable $u = \frac{ds}{dx}$; and in the girders $u = \frac{ds'}{dx}$. The bending moment in the members of the system due to a unit horizontal component of cable stress is m. Calling $T' + M_0 = 0$ and $H = 1$ in the working equation for moment in the girders,

$$m = e' \, (f + a)_m - (y + z) \quad \text{(15)}$$

in all spans. The first term in the denominator of the H-equation (Eq. 14) is:

$$\sum \frac{u^2 \, L}{A \, E} = \sum \int_0^L \left(\frac{ds}{dx}\right)^2 \frac{ds}{E_c \, A_c} + \sum \int_0^L \left(\frac{ds'}{dx}\right)^2 \frac{ds'}{E \, A_g} = \frac{L_s}{E_c \, A_c} + \frac{L_s'}{E \, A_g} \quad \text{(16)}$$

neglecting the effect of change of length of suspenders and tower legs. With a parabolic cable of constant section:

$$L_s = \sum L \, (\sec^3 \alpha + 8 \, n^2 \sec \alpha) \text{ (very nearly)} \quad \text{(17a)}$$

For an eyebar cable in which A_c may be assumed to vary with the cable secant:

$$L_s = \sum L \left(\sec^2 \alpha + \frac{16 \, n^2}{3}\right) \text{ (very nearly)} \quad \text{(17b)}$$

The values of A_c at the low point of each span are used with the value of L_s given in Eq. 17b.[4] In the same manner:

$$L_s' = \sum L \, (\sec^3 \theta + 8 \, c^2 \sec \theta) = \sum L \text{ (very nearly)} \quad \text{(17c)}$$

The second term in the denominator of the H-equation (Eq. 14) is:

$$\sum \int_0^L m^2 \frac{dx}{E \, I} = \sum \int_0^L [e' \, (f + a)_m - (y + z)]^2 \frac{dx}{E \, I}.$$

[4] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), Eqs. 21c and 21d, p. 1145.

Thus for a three-span symmetrical bridge,

$$D = \left(\frac{L_s}{E_c\,A_c} + \frac{L_s'}{E\,A_g}\right) + \frac{(f+a)^2\,L}{3\,E\,I}\left(3\,e^2 - 4\,e + \frac{8}{5}\right)$$
$$+ \frac{2\,L_1}{3\,E\,I_1}\left[e^2\,(f+a)^2 - 2\,e\,(f+a)\,(f_1+a_1) + \frac{8}{5}(f_1+a_1)^2\right] \ldots\ldots (18a)$$

Abbreviating further, Eq. 18*a* may be written also as:

$$D = \frac{(f+a)^2\,L}{3\,E\,I}\left\{\left(\frac{8}{5} + 3\,e^2 - 4\,e\right) + 2\,i\,r\left[\frac{8\,(v')^2}{5} + e^2 - 2\,e\,v'\right]\right.$$
$$\left. + \frac{3\,I}{(f+a)^2\,L}\left(\frac{E}{E_c}\times\frac{L_s}{A_c} + \frac{L_s'}{A_g}\right)\right\} \ldots\ldots (18b)$$

The numerator of Eq. 14 is: $-\sum\int_0^L M'\,m\,\frac{dx}{E\,I} = -\sum\int_0^L (T' + M_0)$

$\times\,[e'\,(f+a)_m - (y+z)]\,\frac{dx}{E\,I}$; but $\sum\int_0^L T'\left[e'\,(f+a)_m - (y+z)\right]\frac{dx}{E\,I} = 0,$

which can be verified by integration. Therefore

$$N = -\sum\int_0^L M_0\,[e'\,(f+a)_m - (y+z)]\,\frac{dx}{E\,I} \ldots\ldots (19)$$

From the foregoing general expression for N (Eq. 19), the value of the numerator N of the H-equation (Eq. 14) for a specific bridge and various loadings may be derived by performing the indicated integration. A three-span symmetrical structure has been assumed in the cases that follow. For a concentrated load P, at a distance $k\,L$ from the left tower in the main span:

$$N = -\,P\,(f+a)\,\frac{k\,L^2}{E\,I}\left[\frac{e}{2}(1-k) - \frac{1}{3}(1 - 2\,k^2 + k^3)\right] \ldots\ldots (20a)$$

and

$$H = \frac{1}{D'\,n'}\left[B(k) - \frac{3}{2}e\,(k - k^2)\right]P \ldots\ldots (20b)$$

in which

$$D' = \frac{3\,E\,I\,D}{(f+a)^2\,L} \ldots\ldots (21a)$$

and

$$B(k) = k\,(1 - 2\,k^2 + k^3) \ldots\ldots (21b)$$

For a concentrated load at a distance $k_1\,L_1$ from the free end of the side span:

$$N = -\,\frac{P\,k_1\,L_1^2}{6\,E\,I_1}\left[e\,(f+a)\,(1-k_1^2) - 2\,(f_1+a_1)\,(1 - 2\,k_1^2 + k_1^3)\right] \ldots (22a)$$

and

$$H = \frac{i\,r^2}{D'\,n'}\left[v'\,B(k_1) - \frac{e}{2}\,(k_1 - k_1^3)\right]P_1 \ldots\ldots (22b)$$

in which

$$B(k_1) = k_1\,(1 - 2\,k_1^2 + k_1^3) \ldots\ldots (23)$$

By a process of integration over proper limits the values of N and H can be obtained from these expressions, for various distributed load conditions. For all spans fully covered with distributed loads of p in the main span and p_1 in the side spans:

$$N = \frac{-p\,L^3}{3\,E\,I}(f+a)\left(\frac{e}{4}-\frac{1}{5}\right) - \frac{p_1\,L_1^3}{3\,E\,I_1}\left[\frac{e}{4}(f+a) - \frac{2}{5}(f_1+a_1)\right] \ldots (24a)$$

and

$$H = \frac{p\,L}{D'\,n'}\left(\frac{1}{5}-\frac{e}{4}\right) + \frac{2\,i\,r^3}{D'\,n'}\left(\frac{v'}{5}-\frac{e}{8}\right)p_1\,L \ldots\ldots\ldots (24b)$$

The value of N for the left or right half of the main span covered with a uniformly distributed load is one half the value for the entire span so loaded. Combined with the value of N for a side span fully loaded, this value is useful in obtaining preliminary values of the maximum moment at the tower and quarter-point of the main span. The value of N for the center 0.4 of the main span covered with a uniformly distributed load, which may be used in obtaining a preliminary value of the maximum moment at the center, is:

$$N = \frac{-p\,L^3}{E\,I}(f+a)\,(0.0473\,e - 0.0390) \ldots\ldots\ldots\ldots (25a)$$

and

$$H = \frac{3\,p\,L}{D'\,n'}(0.0390 - 0.0473\,e) \ldots\ldots\ldots\ldots\ldots (25b)$$

The several formulas given herein, for use in the computation of the live load H of a self-anchored suspension bridge, will all reduce, upon the substitution of f for $(f + a)$, y for $(y + z)$, and $L_s' = 0$, to those of the externally-anchored suspension bridge as derived by others[5,6] for the elastic theory.

Temperature

As the self-anchored suspension bridge is a "closed" structure, no temperature stresses are produced for conditions of uniform temperature change except those caused by the expansion of towers of unequal length below the girders. For the usual case in which the length of the main tower from pier to girder is greater than that at the free end of the side span, the stress produced, however, is merely that in the independent continuous girder as a result of the relative vertical movement of the main tower supports. The moment over the upward deflected supports for a symmetrical three-span girder is:

$$T_1' = T_2' = -\frac{6\,E\,\Delta y_t}{L_1\left(\dfrac{3\,L}{I}+\dfrac{2\,L_1}{I_1}\right)} = -\frac{6\,E\,I\,r\,\Delta y_t}{(3+2\,i\,r)\,L_1^2} \ldots\ldots (26)$$

The corresponding value of H is zero, neglecting the slight effect of horizontal movement of the ends of the girder and the rise of the tower saddles.

---

[5] "A Practical Treatise on Suspension Bridges, Their Design, Construction and Erection," by D. B. Steinman, M. Am. Soc. C. E., 2d Ed., John Wiley & Sons, Inc., New York, N. Y., 1929.

[6] "Modern Framed Structures," by the late J. B. Johnson and C. W. Bryan, Members, Am. Soc. C. E., and F. E. Turneaure, Hon. M. Am. Soc. C. E., 9th Ed., Pt. 2, John Wiley & Sons, Inc., New York, N. Y., 1911.

In the girder hinged at the towers there is no stress produced by the expansion of the towers, since the structure is fully articulated and free to deflect under uniform temperature change without causing stress. Towers fixed at their bases, however, are subject to temperature moments produced by the deflection of the tower tops with the cables.

Live-Load Deflections

The equation of the girder deflection curve may be obtained by performing the double integration of the equation: $\frac{d^2\eta}{dx^2} = -\frac{M}{E\,I}$; from which—

$$\eta = -\int\int\left\{(T' + M_0) + H\left[e'\,(f + a)_m \frac{-4}{L^2}(f + a)\,(L\,x - x^2)\right]\right\}\frac{dx}{E\,I} \quad \text{..(27)}$$

The deflection of the girder at a particular point may be written from this general equation by the substitution of the proper coordinate, or by integration of the familiar equation:

$$\eta' = \int_0^L \frac{M\,m\,dx}{E\,I} \quad \text{........(28)}$$

The change of grade or slope of the deflected girder and roadway is found by integrating Eq. 27 once:

$$\tan\Delta\phi' = \frac{d\eta}{dx} = -\int \frac{M\,dx}{E\,I} \quad \text{........(29)}$$

This slope of the elastic curve of the girder at a particular point is again found by the substitution of the proper coordinate in the general expression. At the towers—$x = 0$, or $x = L$. The formulas that follow have been derived for the three-span symmetrical bridge.

In the main span for a uniformly distributed load extending a distance $k\,L$ from the left tower, the equation of the deflection curve from $x = 0$ to $x = k\,L$ is:

$$E\,I\,\eta = -\left[\frac{T_1'}{2\,L}\left(L\,x^2 - \frac{x^3}{3} - \frac{2\,L^2\,x}{3}\right) + \frac{T_2'}{6\,L}(x^3 - L^2\,x)\right]$$
$$+ H\left[\frac{e}{2}(f + a)\,(L\,x - x^2) + \frac{2}{3}\frac{(f + a)}{L^2}\left(L\,x^3 - \frac{x^4}{2} - \frac{L^3\,x}{2}\right)\right]$$
$$- p\left\{\frac{1}{6}\left[k\,L\left(1 - \frac{k}{2}\right)x^3 - \frac{x^4}{4}\right] - \frac{x\,L^3}{6}\left(\frac{k^4}{4} - k^3 + k^2\right)\right\} \quad \text{....(30)}$$

For the deflection curve from $x = k\,L$ to $x = L$, only the p-terms change. These are: $-p\left[\frac{k^2}{4}L^2\left(x^2 - \frac{x^3}{3\,L}\right) - \frac{x\,L^3}{6}\left(\frac{k^4}{4} + k^2\right) + \frac{k^4\,L^4}{24}\right]$.

At the left tower the change of grade is, for this loading:

$$E\,I\tan\Delta\phi_1' = \frac{L}{3}\left[T_1' + \frac{T_2'}{2} + H\,(f + a)\left(\frac{3\,e}{2} - 1\right)\right]$$
$$+ \frac{1}{6}p\,k^2\,L^3\left(1 - k + \frac{k^2}{4}\right) \quad \text{........(31a)}$$

At the right tower:

$$E\,I \tan \Delta\phi_2' = -\frac{L}{3}\left[\frac{T'}{2} + T_2' + H\,(f + a)\left(\frac{3\,e}{2} - 1\right)\right] + \frac{p\,k^2}{12} L^3 \left(\frac{k^2}{2} - 1\right) \qquad (31b)$$

For full main-span loading and without load on the side spans ($k = 1$, and $T_1' = T_2'$), the deflection at the center of span, where $x = \frac{L}{2}$, is found from the general equation (Eq. 30):

$$\frac{8\,E\,I}{L^2}\Delta f = T_1' + \frac{5}{48} p\,L^2 + H\,(f + a)\left(e - \frac{5}{6}\right) \qquad (32a)$$

The deflection under a concentrated load at the center of the main span may be written from the previous formula by changing simply the deflection coefficient of the independent simple beam:

$$\frac{8\,E\,I}{L^2}\Delta f = T_1' + \frac{P\,L}{6} + H\,(f + a)\left(e - \frac{5}{6}\right) \qquad (32b)$$

The formulas for deflection and load in the side span are most easily obtained by substituting for $T_1 = 0$ in those of the main span. For a uniformly distributed load extending a distance $k_1\,L_1$ from the free end of the span:

$$E\,I_1\,\eta_1 = \frac{-T_1'}{6\,L_1}(x_1^3 - L_1^2\,x_1) + H\left[\frac{-e}{6\,L_1}(f + a)\,(x_1^3 - L_1^2\,x_1) + \frac{2}{3}\left(\frac{f_1 + a_1}{L_1^2}\right)\left(L_1\,x_1^3 - \frac{x_1^4}{2} - \frac{L_1^3\,x_1}{2}\right)\right] - p_1\left\{\frac{1}{6}\left[k_1\,L_1\left(1 - \frac{k_1}{2}\right)x_1^3 - \frac{x_1^4}{4}\right] + \frac{x_1\,L_1^3}{6}\left(\frac{k_1^4}{4} - k_1^3 + k_1^2\right)\right\} \qquad (33a)$$

The change of grade at the free end of the side span is:

$$E\,I_1 \tan \Delta\phi_1' = \frac{L_1}{3}\left\{\frac{T_1'}{2} + H\left[\frac{e}{2}(f + a) - (f_1 + a_1)\right]\right\} + \frac{p_1\,k_1^2}{6} L_1^3 \left(1 - k_1 + \frac{k_1^2}{4}\right) \qquad (33b)$$

At the tower:

$$E\,I_1 \tan \Delta\phi_1' = -\frac{L_1}{3}\left\{T_1' + H\left[e\,(f + a) - (f_1 + a_1)\right]\right\} + \frac{p_1}{12} k_1^2\,L_1^3 \left(\frac{k_1^2}{2} - 1\right) \qquad (33c)$$

The deflection at the center of the fully loaded side span is:

$$\frac{16\,E\,I_1}{L_1^2}\Delta f_1 = T_1' + \frac{5}{24} p_1\,L_1^2 + H\left[e\,(f + a) - \frac{5}{3}(f_1 + a_1)\right] \qquad (33d)$$

For a central concentrated load, as in the main span, $\frac{P_1 L_1}{3}$ should be written in place of $\frac{5}{24} p_1 L_1^2$ in Eq. 33*d*. Eqs. 27 to 33, with the *p*-terms made equal to zero, are those for the deflection of spans unloaded themselves, which are influenced by the loading in other spans through the action of girder continuity and cable effect. For girders hinged at the points of support, the continuity terms are made equal to zero in these and all formulas previously derived for continuous girders. For bridges with unloaded side-span cables, f_1 is made equal to zero in Eqs. 3 to 33.

STRESSES

The formula for cable stress is the same as that for the externally-anchored bridge. Therefore the stress at any point in the cable from the combination of dead and live load is $(H_w + H) \sec \phi$, in which the angle ϕ is defined by $\tan \phi = \tan \alpha + \frac{4 f}{L^2} (L - 2 x)$. The influence of deflection on the cable angle ϕ, as well as on the girder angle ϕ', is negligible in its effect on both cable and girder stresses.

Since the girder, in helping to carry the dead load of the bridge, is ordinarily subject to no moment or normal shear, the only dead-load stress that it must be designed to sustain is the direct compression that amounts to $H_w \sec \phi'$ at any point. This stress is comparable to the dead-load stress in the cable.

Live-load girder stresses are those of direct compression, moment, and shear. The stresses from direct compression and moment about the axis of the girder may be treated, as in the arch, by finding, at each section examined along the girder, the combined compressive stress and the resultant tensile stress corresponding to the moments alone about the proper kern points, which are distant $\frac{r^2}{c_{U,L}}$ from the neutral axis.[6] The final flange stresses are $f_{U,L} = M_{U,L} \frac{c_{U,L}}{I}$. Therefore, influence lines for flange stress may be obtained by determining the values of the moments about the kern points for unit live-load concentrations along the bridge. This method of obtaining influence lines for girder stress is used by some German writers.[7]

The kern moments may be determined either analytically or graphically. Analytically, they are equal to the moment previously found about the geometrical girder axis (on which lie the cable reaction pins at the span ends) corrected for the effect of the components of H and V acting through the lever arms $\frac{r^2}{c_{U,L}} \mp h$. Thus in Fig. 5:

$$M_{U,L} = M \pm [V \sin \phi' + H \cos \phi'] \left(\frac{r^2}{c_{U,L}} \mp h \right) \quad \ldots\ldots\ldots (34)$$

in which

$$\phi' = \tan^{-1} \left[\tan \theta + \frac{4 a}{L^2} (L - 2 x) \right] \quad \ldots\ldots\ldots\ldots\ldots (35)$$

The kern points shown in Fig. 5 for the upper and lower flanges are $K_{U,L}$.

[7] "The Second Fixed Bridge Over the Rhine at Cologne," by W. Dietz, *Zeitschrift des Vereines Deutscher Ingenieure*, 1920.

Eq. 34 might also be written:

$$M_{U,L} = (T' + M_0)_{U,L} + H\,[e'\,(f + a)_m - (y + z)]_{U,L}$$
$$\pm H \left(\frac{r^2}{c_{U,L}} \mp h \right) \cos \phi' \ldots\ldots\ldots\ldots\ldots (36)$$

The influence of the component of V in Eq. 34 is included in Eq. 35 by writing the values of $(T' + M_0)$ and $H\,[e'\,(f + a)_m - (y + z)]$ directly with respect

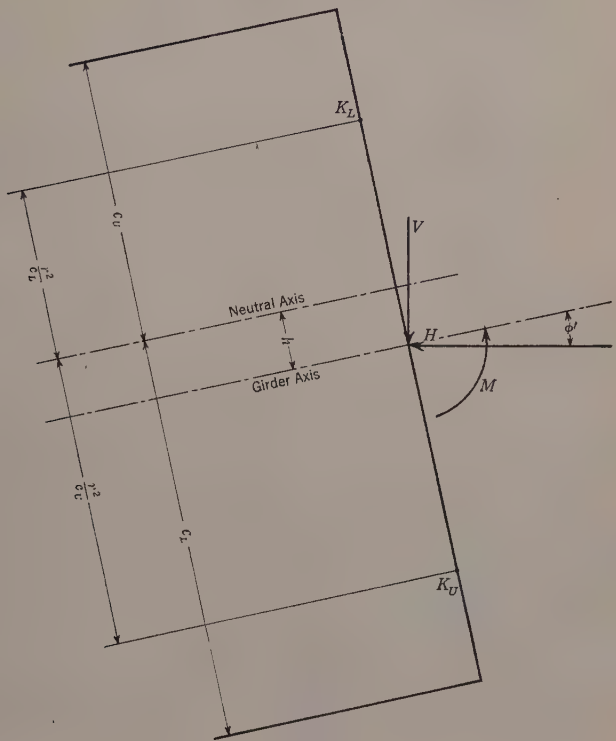


Fig. 5

to the kern points; or, in other words, as in the definition, by taking moments directly about the kern points. This can be done more easily graphically than analytically, however. Fig. 6 shows the method of determining graphically the value of $\frac{M_{U,L}}{H}$ at any section for a unit load at a given point. The equilibrium polygons for $-(y + z)$, $e'\,(f + a)_m$, $\frac{T'}{H}$, and $\frac{M_0}{H}$ are added graphically, and the $\frac{M_{U,L}}{H}$-intercept can then be scaled from points distant $\pm \left(\frac{r^2}{c} \mp h \right) \cos \phi'$, from points on the $(y + z)$-line directly over or under the kern points of the girder. The case given in Fig. 6 is that of a symmetrical three-span bridge, with a girder (symmetrical top and bottom) that lies on a straight grade line

in the side spans, loaded with concentrated loads in the main span and left side span. It will be noted that it is necessary to redraw only the $\frac{T'}{H}$-polygon and $\frac{M_0}{H}$-polygon for the various loads, since the other polygons remain unchanged.

Ordinarily, the shearing stresses in the girder will be small, and therefore it should be necessary to determine the shear merely at a few critical sections to test the adequacy of the web, to estimate the maximum permissible rivet and stiffener spacing, and to compute the girder reactions. To draw the influence

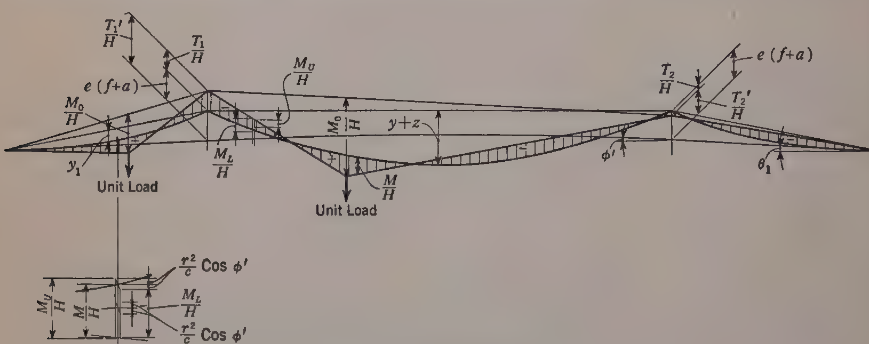


FIG. 6.—EQUILIBRIUM POLYGONS

lines for shear at these few sections it will be more advantageous to draw them directly than to construct $\frac{V}{H}$-polygons for each load position similar to the method used in obtaining the stress-moment influence-line ordinates. The latter can also be obtained directly with advantage if only a few sections are to be investigated.

The normal shear in the girder is:

$$V_N = V \cos \phi' - H \sin \phi' \qquad (37)$$

in which V, the vertical shear, is written as in Eq. 13, and therefore:

$$V_N = \left(\frac{T_2' - T_1'}{L} + V_0 \right) \cos \phi'$$

$$- H \left[4 \left(\frac{f + a}{L^2} \right) (L - 2x) - (f + a)_m \left(\frac{e_2 - e_1}{L} \right) + \tan \phi' \right] \cos \phi' \qquad (38a)$$

To construct the influence line, this equation may be rewritten in the following form:

$$\frac{V_N \sec \phi'}{\left[4 \left(\frac{f + a}{L^2} \right) (L - 2x) - (f + a)_m \left(\frac{e_2 - e_1}{L} \right) + \tan \phi' \right]}$$

$$= \frac{\left(\frac{T_2' - T_1'}{L} + V_0 \right)}{\left[4 \left(\frac{f + a}{L^2} \right) (L - 2x) - (f + a)_m \left(\frac{e_2 - e_1}{L} \right) + \tan \phi' \right]} - H \qquad (38b)$$

Effect of Motion of Ends of Girder Due to Deflection of Bridge

Due, principally, to vertical deflection in the main spans, the ends of the girder will move horizontally to produce a change in H from that evaluated by the elastic theory. This change in H can be determined best by successive trials from the values of the main-span deflection calculated for the new values of H. The change in H may be written

$$\Delta H = \pm \frac{\Delta x}{D} \qquad (39a)$$

in which $\pm \Delta x$ is the entire change in horizontal length of girder.

$$\Delta H = \pm \frac{3 \Delta x E I}{(f + a)^2 D' L} \qquad (39b)$$

In some cases this correction in H amounts to several per cent—with corresponding percentages of correction in moment. The moment corrections will be greater for load in the main span and less for load in the side span, but generally are not great enough to affect the design, although they should be considered in analyzing the motion of the structure during erection.

Effect of Variable Girder Moment of Inertia

Although the integrations required for the determination of N and D in the H-formula (Eq. 14) may be performed for varying girder moment of inertia, as in arch analysis, the value of H is usually little affected by variation in I within the span. The values of the moments, of course, are more greatly affected by the variation in I. To account for this effect the values of T' and e' corresponding to such variation can be computed by the moment-area or a comparable method and the results used in combination with the H obtained for constant I. Where constant-depth girders are used in short spans, the moment of inertia will tend to be fairly constant since it is impracticable actually to follow the moment curve closely in selecting the girder sections; and therefore the assumption of a uniform average I within the span will usually yield sufficiently accurate results.

Effect of Suspender Elongation

Analysis by the usual procedure to consider the effect of suspender elongation indicates that this action produces a negligible change in the values of moment, shear, and deflection of the girder from those computed neglecting this effect.

Unsymmetrical and Multiple Spans

The formulas for unsymmetrical three-span or multiple-span bridges require different values of the continuity terms e' and T' from those of the symmetrical three-span bridge. The changed values of e' and T' result in modification of N and D in the H-equation and of the expressions for shear, moment, and deflection. The formulas for dead load, however, are unchanged. The form of all the equations for bridges with girders hinged at the towers remains

unchanged from the form of the equations of the three-span symmetrical bridge, since the girder continuity terms e' and T' become zero for the hinged condition.

Computations for a Self-Anchored Suspension Bridge with a Continuous Stiffening Girder

A bridge with a 20-ft roadway, two 2-ft walkways, and a lightweight floor designed for H-15 highway loading is examined. The structure is symmetrical, with a 350-ft main span and two 150-ft side spans on 5% grades. The dead-load cable sag is 35 ft, and the parabolic camber of the main-span floor and girder is 4.38 ft. The open cable is made up of nine twisted wire strands, and single 36-in. wide flange beams are used for the stiffening girder.

Span and Load Constants.—Let $L = 350$ ft; $L_1 = 150$ ft; $f = 35$ ft; $n = 0.10$; $a = 4.38$ ft; and $n' = 0.1125$. The dead loads are: $w = 1{,}330$ lb per ft, and $w_1 = 1{,}430$ lb per ft; and the live load is $p = 500$ lb per ft (all loads are for one girder). For the cable, $E_c = 25{,}000{,}000$ lb per sq in.; and, for the girder, $E = 29{,}000{,}000$ lb per sq in. The dead load (see Eq. 1*b*) is: $H_w = \frac{1{,}330 \times 350^2}{8(35 + 4.38)} = 517$ kips; the side-span cable sag (Eq. 4*a*) is $f_1 = \frac{1{,}430 \times 150^2}{1{,}330 \times 350^2} \times 39.38 = 7.82$ ft; $L_s = 743$ ft; $L_s' = 651$ ft; $I = 15{,}625$ in.^{4}; $I_1 = 21{,}175$ in.^{4}; the area of the cable is $A_c = 12.1$ sq in.; of the main-span girder, $A_g = 70.3$ sq in.; of the side-span girder, $A_{g1} = 86.1$ sq in.; $i = 0.738$; $r = 0.429$; $v' = 0.199$; $e = 0.585$; and $D' = 0.444$.

Live Load—Spans Fully Loaded.—With the main span fully loaded (Eq. 24*b*) $H = \frac{500 \times 350}{0.444 \times 0.1125}\left(0.20 - \frac{0.585}{4}\right) = 189$ kips; and $T_1' = \frac{-500 \times 350^2}{4 \times 3.632} = -4{,}215$ kip-ft. By Eq. 32*a*: $\frac{8 \times 29{,}000 \times 15{,}625}{350^2 \times 144} \times \Delta f = -4{,}215 + \frac{5}{48} \times 500\,(350)^2 + 189 \times 39.38\,(0.585 - 0.833)$; or $\Delta f = +1.58$ ft. The live load camber equals $a - \Delta f = 2.80$ ft.

To obtain the correction in H due to the movement of the ends of the girder corresponding to the change in camber: $\Delta L' = \frac{8}{3}(4.38^2 - 2.80^2)\frac{1}{350} = +0.087$ as a parabola; and $\frac{16 \times 29{,}000 \times 21{,}175}{150^2 \times 144}\Delta f_1 = -4{,}215 + 189\,(0.585 \times 39.38 - 1.667 \times 7.82)$. That is, $\Delta f_1 = -0.77$ ft; $\Delta L' = \frac{8}{3}(0.77)^2 \times \frac{2}{150} = -0.020$ ft; total $\Delta L' = +0.087 - 0.020 = +0.067 = \Delta L$ (very nearly); and (Eq. 39*b*) $\Delta H = \frac{+3 \times 0.067 \times 29{,}000 \times 15{,}625}{39.38^2 \times 350 \times 0.444 \times 144} = 2.6$ kips $= 1.4\%$. Furthermore, $H' = 188.5 + 2.5 = 191$ kips; $\Delta f' = +1.48$ ft; $\Delta f_1' = -0.76$ ft; the maximum stress in the side-span cable is $(517 + 191)\,1.147 = 813$ kips; the maximum direct stress in the girder is $(517 + 191)\,1.001 = 709$ kips; and (Eq. 12) the negative moment at the center of side span is: $M = -2{,}108 + 191 \times \left(\frac{0.585}{2} \times 39.38 - 7.82\right) = -1{,}400$ kip-ft. At the kern point: $M = -1{,}400$

-191×1.04 ft $= -1{,}600$ kip-ft. It will be noted that the effect, on the moment, of the correction in H is not important.

With the side span fully loaded (Eq. 24b): $H = \frac{0.738 \times 0.429^3}{0.444 \times 0.109} \times \left(\frac{0.199}{5} - \frac{0.585}{8}\right) 500 \times 350 = -7.0$ kips; $T_2' = \frac{+0.738 \times 0.429^3}{3.632 \times 1.632} \times \frac{500 \times 350^2}{4} = +150$ kip-ft; and $T_1' = -395$ kip-ft.

The positive moment at the center of the side span is $M = -\frac{395}{2} + \frac{500 \times 150^2}{8} - 7.0\left(\frac{0.585}{2} \times 39.38 - 7.82 \mp 1.04\right) = +1{,}190$ kip-ft; and, with far side span loaded: $M = \frac{+150}{2} - 7.0\left(\frac{0.585}{2} \times 39.38 - 7.82 \mp 1.04\right) = +55$ kip-ft.

Influence Line Solution for Live-Load Girder Flange Stress.—The influence-line values for H are as follows: With a load in the main span—

$$H = \frac{1}{0.444 \times 0.1125}\left[B(k) - \frac{3}{2} \times 0.585\, k\,(1 - k)\right] \ldots\ldots (40a)$$

and with a load in the side span—

$$H = \frac{0.738\,(0.429)^2}{0.444 \times 0.1125}\left[0.199\, B(k_1) - \frac{0.585}{2} k_1\,(1 - k_1^2)\right] \ldots (40b)$$

TABLE 1.—COMPUTATIONS FOR INFLUENCE LINES

(a) H					(b) M_0 AND T'						
k and k'	Y	Z	$Y - Z$	H	M_0	$\frac{M_0}{H}$	T_1'	$\frac{T_1'}{H}$	T_2'	$\frac{T_2'}{H}$
MAIN-SPAN COMPUTATIONS											
0.1	0.0981	0.0790	0.0191	0.382	31.5	82.5	−20.7	−54.1	− 5.3	−13.9
0.2	0.1856	0.1404	0.0452	0.905	56.0	61.9	−33.4	−36.9	−12.8	−14.2
0.3	0.2541	0.1843	0.0698	1.397	73.5	52.6	−39.3	−28.2	−21.3	−15.3
0.4	0.2976	0.2106	0.0870	1.742	84.0	48.2	−39.9	−22.9	−29.5	−16.9
0.5	0.3125	0.2194	0.0931	1.864	87.5	47.0	−36.1	−19.4	−36.1	−19.4
SIDE-SPAN COMPUTATIONS											
0.2	0.0369	0.0562	−0.0193	−0.0525	24.0	−457	− 4.03	76.8	+1.53	−29.1
0.4	0.0591	0.0983	−0.0392	−0.1060	36.0	−340	− 7.05	66.5	+2.68	−25.3
0.6	0.0591	0.1123	−0.0532	−0.1440	36.0	−250	− 8.05	55.9	+3.06	−21.2
0.8	0.0369	0.0842	−0.0473	−0.1280	24.0	−187.5	− 6.05	47.2	+2.30	−18.0

The computations are given in Table 1. To simplify typography in Table 1(a), Eqs. 40 may be written

$$H = X\,(Y - Z) \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

the proper quantities being substituted from Eq. 40a in the case of main-span computations and Eq. 40b in the case of side-span computations.

As previously explained the value of $e' (f + a) m - (y + z)$ is laid out from the center line of the girder on the elevation of the bridge drawn to scale. The values of $\frac{T' + M_0}{H}$ are then plotted to the same scale for each unit load, and the intercepts between these two lines at points along the span are measured from points distant $\pm \frac{r^2}{c} \cos \phi'$ from the $(y + z)$-line. The product of these intercepts by the values of H for the respective loads are flange-stress moments for live load. Adding these moments the maximum total positive and negative flange-stress moments are obtained. The maximum live-load flange-stress moments are:

Side Span		Main Span	
Point	Stress moment (kip-ft)	Point	Stress moment (kip-ft)
0	0	1.0 (tower)	−1,190
0.2	−1,120	0.1	− 680
0.4	−1,560	0.2	+ 930
0.6	−1,525	0.3	+1,045
0.8	−1,040	0.4	+ 910
		0.5	+ 735

The total live-load flange stresses for both direct and bending stress are then obtained by dividing the foregoing moments by the section modulus of the symmetrical girder at the several points. The dead-load stress, which is direct, is added to the live-load stress to obtain the total compressive flange stress for dead and live loads.

Live-Load Shear.—The maximum live-load shears at the ends of the main and side spans may be obtained by constructing influence diagrams that consist of graphically combining the influence lines for: V_0; $\left(\frac{T_2' - T_1'}{L}\right)$; and $-H\left[4\frac{(f+a)}{L^2}(L - 2x) - (f + a)_m\left(\frac{e_2 - e_1}{L}\right) + \tan\phi'\right]$. In obtaining vertical shear, $\tan \phi'$ under the bracket is omitted. Proceeding in this manner the following values are obtained:

Span	Vertical shear (kips)	Normal shear (kips)
End of main span..........	+36.2, −28.6	+36.8, −36.0
Tower end of side span.......	−44.1, +41.0	−43.8, +31.8
Free end of side span........	+36.6, −39.3	+37.4, −48.1

The end shears were all determined for full-span loading except for the main-span shear from main-span loading and normal shear at the tower end of the side span for main-span loading, which was slightly increased for partial main-span loading. The influence diagram for shear at the end of the main span is shown in Fig. 7.

The lateral wind stresses in the stiffening girders for spans such as these may be neglected since they amount to less than 25% of the total dead-load plus live-load stresses. In general, the wind stresses in self-anchored suspension

bridges are relatively unimportant, since span lengths are moderate, elastic theory design of the girders requires that they be comparatively stiff, and the girders must be designed to carry dead and live load thrust in addition to moment.

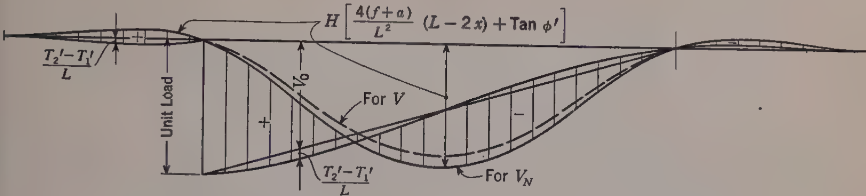


FIG. 7.—INFLUENCE DIAGRAM FOR SHEAR AT END OF MAIN SPAN

ACKNOWLEDGMENT

This paper was submitted by the author, in 1940, as a thesis to the graduate faculty of Rutgers University at New Brunswick, N. J., in partial fulfilment of the requirements for the degree of Civil Engineer.

CONCLUSION

The theory of the self-anchored suspension bridge has been presented in the foregoing in such a manner as to enable the practicing engineer readily to apply the formulas derived to the design of that type of structure. It is evident that the self-anchored suspension bridge has many advantages over other types in certain locations; and, with the theory for its design conveniently at hand, the self-anchored suspension bridge of moderate span should become widely adopted, where conditions justify its use.

APPENDIX

NOTATION

For the most part the notation in this paper (see Fig. 1) is the same as that adopted by D. B. Steinman.[5,8] An effort is made, furthermore, to conform substantially with American Standard Symbols for Mechanics, Structural Engineering, and Testing Materials compiled by a Committee of the American Standards Association[9] with Society representation, and approved by the Association in 1932.

Throughout, a subscript 1 is used to distinguish side-span symbols from main-span symbols; subscript U = "upper kern point" and L = "lower kern point"; and w = "due to dead load."

A = area of cross section, the subscript c denoting "cable" and g denoting "stiffening girder";

a = center camber of stiffening girder;

[8] "A Generalized Deflection Theory for Suspension Bridges," by D. B. Steinman, *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 1135.

[9] A.S.A.—Z10a—1932.

c = distance from the extreme fiber to the neutral axis;

D = general form of the denominator of the H-equation (Eqs. 14 and 18):

D' = working form (see Eq. 21a);

E = modulus of elasticity of girder:

E_c = elastic modulus of cable;

e = continuity factor for suspender and arch action (Eq. 10):

e' = general form of continuity factor;

e_1 = continuity factor at tower 1;

e_2 = continuity factor at tower 2;

f = cable sag at the center of the main span;

H = horizontal tension in cable and compression in girder due to live load p:

H_w = horizontal tension H due to dead load;

I = moment of inertia of stiffening girder;

i = ratio of moments of inertia, main span to side span, $\frac{I}{I_1}$;

K = a kern point;

k = ratio of span length:

$k\,L$ = a span segment;

L = main-span length:

L_s = cable length function;

L_s' = girder length function—functions defined by Eq. 17;

M = resultant bending moment at any section x of the stiffening girder:

M' = the moment in the members of the system with the cable cut;

M_w' = dead-load moment carried by the girder, as a parabolic arch;

M_0 = simple-beam bending moment due to a live load p;

M_D = simple-beam bending moment due to dead load w;

m = bending moment in the members of the system due to a unit horizontal component of cable stress;

N = numerator of H-equation (Eq. 14);

n = ratio of cable sag to span length, $\frac{f}{L}$:

$n' = \frac{f + a}{L}$;

P = concentrated load;

p = live load per unit length:

p_g = part of the live load taken by the girder in moment;

r = ratio of length of side span to main span, $\frac{L_1}{L}$;

S = direct stress in the various members of the system:

S' = direct stress in the various members of the system with the cable cut;

s = length of cable:

ds = increment of cable length;

T = bending moment at any section x due to continuity:

- T_1 = moment at tower 1;
- T_2 = moment at tower 2;
- T' = moment effect of continuity on the independent girder (cable removed); general form;
- T_1' = T' at tower 1;
- T_2' = T' at tower 2;

t = temperature;

u = direct stress in the members of the system, corresponding to a horizontal component of stress equal to unity in the cable and girder;

V = total vertical shear at any section x of the stiffening girder, due to live load p:

- V_N = normal shear;
- V_0 = simple-beam vertical shear due to live load;

$v = \dfrac{f_1}{f}$:

- $v' = \dfrac{f_1 + a_1}{f + a}$;

w = uniformly distributed dead load:

- w_a = part of the dead load carried by the stiffening girder as an "arch";

X = a substitution factor in Eq. 41;

x = abscissas in main span measured from the tower:

- x_1 = abscissas in the side span measured from the free end of the side span;

Y = a substitution factor in Eq. 41;

y = cable ordinate;

Z = substitution factor in Eq. 41;

z = stiffening girder ordinate;

α = slope of cable chord in any span;

Δ = increment:

- Δf = midspan deflections;
- Δy_t = vertical deflection of the girder at the towers due to temperature change;
- $\Delta\phi$ = angular deflection of cable;
- $\Delta\phi'$ = angular deflection of girder;
- Δx = change in horizontal length of girder;

η = deflection of truss at any section x;

θ = slope of the girder chord in any span;

ϕ = slope of cable at any point:

- ϕ' = slope of the girder at any point.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

ON THE METHOD OF COMPLEMENTARY ENERGY

AND ITS APPLICATION TO STRUCTURES STRESSED BEYOND THE PROPORTIONAL LIMIT, TO BUCKLING AND VIBRATIONS, AND TO SUSPENSION BRIDGES

BY H. M. WESTERGAARD,[1] M. AM. SOC. C. E.

SYNOPSIS

The method of complementary energy is a general method of structural mechanics. The basic law was stated by F. Engesser in a paper in 1889. He extended Castigliano's law of least work to apply beyond the range of Hooke's law by replacing work by complementary work, which is an integral of distance times increment of force. Engesser's paper is little known.

The purpose of the present paper is to give proof and demonstration of the method. The proof goes back to fundamentals and includes a re-examination of the fundamentals; this is needed to remove doubts about the ranges of applicability. The demonstration consists of representative applications and may be interpreted as an exploration of the field.

HISTORICAL NOTES

The method of complementary energy is an extension of Castigliano's method of least work. Alberto Castigliano[2] published his method during the Seventies in papers and a treatise. His principle of least work applies to statically indeterminate structures stressed within the range of Hooke's law and subject to the restriction that all significant deformations must be linear homogeneous functions of the loads. Castigliano showed that among all the

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by June 15, 1941.

[1] Dean, Graduate School of Eng., and Gordon McKay Prof. of Civ. Eng., Harvard Univ., Cambridge, Mass.

[2] Thesis to obtain diploma as engineer, Torino, 1873; two papers in *Atti della Reale Accademia delle Scienze di Torino*, Vol. 10, 1875, p. 380, and Vol. 11, 1876, p. 127; and "Théorie de l'équilibre des systèmes élastiques," Torino, 1879, 480 pp., translated into English by E. S. Andrews under the title "Elastic Stresses in Structures," Scott, Greenwood & Son, London, 1919.

statically possible states of stress in such a structure the correct one is that which makes the energy of the internal stresses a minimum. This state of stress satisfies automatically not only the requirement of equilibrium but also the requirement of geometrical continuity. L. F. Ménabréa[3] had stated this principle clearly for trusses in 1858, but his proof contained misunderstandings, and the method is credited justly to Castigliano.

Castigliano[4] himself gave the method its first extension; he stated a revised expression that must be made a minimum if imperfect fits of redundant members create initial stresses; and he applied this procedure to temperature stresses in a general discussion and in six examples.[5] H. Müller-Breslau[6] improved the procedure for temperature stresses and contributed much toward making Castigliano's method known. A useful and dependable critical account of the original works in the field was given by M. Grüning[7] in 1912. In the twentieth century Castigliano's method has become stock in trade; it holds a position today as one of several useful general procedures of structural mechanics. It is worthy of note that in a book published in 1936 R. V. Southwell[8] of Oxford University, Oxford, England, gave an attractive original derivation of Castigliano's principle, based on a discussion of self-strains.

The contribution that has the greatest interest for the present study was published by Fr. Engesser[9] in a paper in 1889. He derived the basic law of the method of complementary energy. It is a modification of Castigliano's law of least work in which work is replaced by complementary work or complementary energy. As work is an integral of force times increment of distance or of stress times increment of deformation, so is complementary work an integral of distance times increment of force or of deformation times increment of stress. Engesser's theory applies beyond the range of Hooke's law; it includes not only Castigliano's method but also Müller-Breslau's procedure for temperature stresses as special applications. In his review of the field in 1912 Grüning[10] quoted and discussed Engesser's contribution, but otherwise it has received little attention. A plausible explanation is that structural analysis has been concerned mainly with stresses below the proportional limit, and the applicability to buckling and vibrations had not been realized.

The method of complementary work or complementary energy is analogous to another method which has become important in structural statics; namely, the method based on the "principle of minimum of the potential energy by variation of the shape." It is advantageous to consider the two methods in

[3] "Nouveau principe sur la distribution des tensions dans les systèmes élastiques," by L. F. Ménabréa, *Comptes Rendus*, Paris, Vol. 46, 1858, pp. 1056–1060.

[4] "Théorie de l'équilibre des systèmes élastiques et ses applications," Torino, 1879, p. 39.

[5] *Loc. cit.*, pp. 39, 317, 324, 332, 347, 428, and 442.

[6] "Der Satz von der Abgeleiteten der ideellen Formänderungs-Arbeit," by H. Müller-Breslau, *Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover*, Vol. 30, 1884, columns 211–214; "Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktionen," Leipzig, 1886, 5th Ed., 1924, pp. 74–79; "Graphische Statik der Baukonstruktionen," Vol. 2, Subvolume 1, Leipzig, 1892, p. 49, 5th Ed., 1922, p. 47.

[7] "Theorie der Baukonstruktionen I: Allgemeine Theorie des Fachwerks und der vollwandigen Systeme," by M. Grüning, *Encyklopädie der mathematischen Wissenschaften*, Vol. 4, Subvolume 4, Leipzig, 1907–1914, pp. 419–534, especially pp. 437–454.

[8] "An Introduction to the Theory of Elasticity," by R. V. Southwell, Oxford Univ. Press, 1936, p. 91.

[9] "Ueber statisch unbestimmte Träger bei beliebigem Formänderungs-Gesetze und über den Satz von der kleinsten Ergänzungsarbeit," by Fr. Engesser, *Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover*, Vol. 35, 1889, columns 733–744, especially 738–744.

[10] *Loc. cit.*, p. 454.

conjunction. The basic principle of the second method is sometimes stated as a direct consequence of a general law of dynamics, but it can be derived from the simplest laws of statics. Daniel Bernouilli and Leonhard Euler[11] stated and used this principle in a special form in the first half of the eighteenth century. The method played a part in the nineteenth century. A mathematical paper published by W. Ritz[12] in 1908 gave new impetus to its use, which has been widespread since then. A number of applications are found in the writings of S. Timoshenko.[13]

The Laws of Least Energy and of Least Complementary Energy

Since it is desirable that no doubts shall remain about the ranges of applicability, all the steps of the derivations of the basic theorems will be shown.

The Structure.—Fig. 1 represents a structure of general type. The black parts are joints, which, by a definition adopted here, are rigid bodies. The shaded parts are deformable members, which are attached to the joints. External forces and reactions are assumed to act on joints only. Since the joints may be three-, two-, or one-dimensional or may be without extension, and any number of joints may be assumed to exist, it is difficult to conceive of a structure to which this picture could not be adapted.

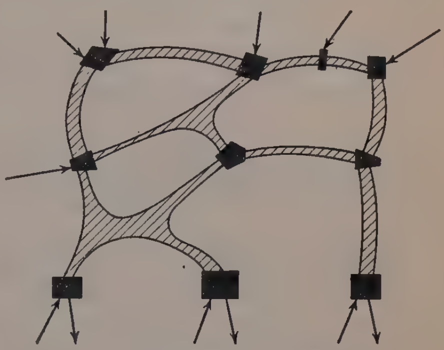


Fig. 1.—Structure of General Type

Notation.—The notation applied to the structure in Fig. 1 requires explanations, which follow:

P = external load. The load P may be a single force P, or it may consist of a group of forces $c_1 P$, $c_2 P$, $c_3 P$, $\cdots$, all proportional to P and varying together when P varies. This conception makes it possible to consider a whole dead load or a whole live load as "a load"; which is in accordance with common usage of the word "load."

p = path of the load P during the deformation from a starting shape of the structure to an assumed shape. If P is a single constant force, p is the displacement of the point of application in the direction of P, and Pp is the work of P. If the load P is a group of forces, then p is defined by the statement that with the value of P remaining constant during the assumed deformation, the work of P is Pp. For example, if P is a total load uniformly distributed over the length of a beam, p is the increase of the average deflection.

S = stress in a member. The stress in the sense adopted here is interpreted as a load exerted by joints on a member, and is defined in the same way

[11] "De curvis elasticis," by Leonhard Euler, 1744; annotated translation into English by W. A. Oldfather, C. A. Ellis, M. Am. Soc. C. E., and Donald M. Brown, in *Isis*, Vol. 20, 1, November, 1933, pp. 72–160, especially p. 78.

[12] *Crelles Journal*, Vol. 135, 1908.

[13] See especially "Theory of Elasticity," 1934, "Theory of Elastic Stability," 1936, and "Theory of Plates and Shells," 1940, by S. Timoshenko, McGraw-Hill Book Co., Inc.

as P except for the restriction that all the forces constituting a particular stress S must be in equilibrium. If the member is a simple tension member in a truss, S may be taken as the total tension, and the load on the member is the two equal and opposite pulls, each equal to S, at the ends. There can be more than one stress in a member. For example, in a beam flat joints may be assumed at two adjacent cross sections a distance dx apart. The member between these two joints has two stresses: One is the bending moment M, which consists of two equal and opposite couples exerted by the joints; the other is the transverse shear V, which consists of two equal and opposite shearing forces and a supplementary balancing couple Vdx at one of the joints. Instead one may in this case assume two members occupying the same space between the cross sections, one resisting M and the other V.

D = deformation in the direction of S; D is the path of S, defined in terms of S as p in terms of P. In the simple tension member with total tension S, D is the total elongation. In the member in the beam D in the direction of M is the relative rotation of the two joints, that is, $-\frac{d^2y}{dx^2}\,dx$ if y denotes the deflections due to the bending moments, y being measured from the starting shape; and D in the direction of V is the relative sliding of the joints.

R = reaction; a force or group of forces exerted by a support, otherwise defined as P.

r = path of R; displacement of a support in the direction or R; $-r$ is the settlement of the support against R.

Equation of Virtual Work in Infinitesimal Form.—The structure is in equilibrium if all members and all joints are in equilibrium. The members are loaded only by the stresses S, and are automatically in equilibrium by the definition of the stresses. A joint, by definition, is a rigid body. The equilibrium of a rigid body can be investigated by assuming an arbitrary infinitesimal movement of it while the forces remain constant. If during any such movement the sum of work of all the forces is zero except for an infinitesimal quantity of second or higher order, the rigid body is in equilibrium. Assume an arbitrary infinitesimal movement of each joint, whereby the deformations p, D, and r increase by the amounts δp, δD, and δr. The forces acting on all the joints are the forces constituting the loads P, the stresses reversed or $-S$, and the reactions R. The condition that the sum of work of all the forces on all the joints must be zero is expressed by the equation

$$\sum P\,\delta p - \sum S\,\delta D + \sum R\,\delta r = 0 \qquad (1)$$

This equation will be recognized as the equation of virtual work in infinitesimal form; it states a principle that has played a great part in mechanics. Eq. 1 is a general condition of equilibrium, unrestricted by any requirement of linearity of the relations between the deformations, stresses, and loads; that is, unrestricted by any law of superposition.

Initial Assumption.—Changes of temperature can be considered to have taken place in advance, before the forming of the starting state. Then it becomes feasible to assume, and it will be assumed, that in all operations that

need be considered in the analysis each stress S is a definite continuous function of the deformations D of the member in which it belongs, and conversely each deformation is a definite continuous function of the stresses in the member.

Energy.—Now consider a series of changes of shape, a variation of the shape, during which the loads and the positions of the supports remain constant; that is, the values of P and r remain constant. Each joint is moved from the starting position into an arbitrarily assumed position; thereby the structure is changed from the starting shape to an assumed shape. The members remain attached to the joints, so that continuity is maintained, but the requirements of equilibrium are ignored in this operation; that is, the geometrical requirements are respected, while the statical requirements are abandoned temporarily. The stresses S will be the proper definite functions of the deformations D. Then, for each assumed shape the structure and the loads can be said to have a potential energy equal to

$$T = \sum \int_{D_0}^{D} S\, dD - \sum P\, p \qquad (2)$$

in which the lower limits D_0 are chosen arbitrarily and the summations include all stresses and loads. The phrase "potential energy" can be used here because the stresses are definite functions of the deformations in the operations of the analysis. An infinitesimal variation of the assumed shape causes an increment, a first variation, of the potential energy equal to

$$\delta T = \sum S\, \delta D - \sum P\, \delta p \qquad (3)$$

Let the requirements of equilibrium be imposed again. Then Eq. 1 must be satisfied. Since $\delta r = 0$ in the variation considered, it follows that

$$\delta T = 0 \qquad (4)$$

That is, if not only all the geometrical requirements but also all the statical requirements are to be satisfied, the first variation of the potential energy must be zero; that is, the potential energy must be a minimum or a maximum. Further arguments, omitted here, show that a stable equilibrium requires that T be a minimum. The statement

$$T = \min \qquad (5)$$

is interpreted then with the reservation that T is normally a minimum, but under some circumstances the minimum may be replaced by a maximum. Eqs. 5 and 2 together state the "law of minimum of the potential energy by variation of the shape." Eq. 5 like Eq. 1 is unrestricted by any law of superposition.

Assumption of Superposition of Deformations, and the Equation of Virtual Work in Finite Form.—It will now be assumed that the law of superposition (or linearity) applies in a limited form; namely to deformations only, without involving loads, stresses, and reactions: it is assumed that if p', D', r' and p'', D'', r'' are two geometrically possible sets of deformations p, D, r, then $p'+p''$, $D'+D''$, $r'+r''$ are a possible set, provided that all the deformations involved are within some range. If this law is assumed, δp, δD, and δr in Eq. 1 may be

replaced by p', D', and r'; so that

$$\sum P\,p' - \sum S\,D' + \sum R\,r' = 0 \qquad (6)$$

This is the equation of virtual work in finite form, which has been important in structural mechanics since the Seventies, when Otto Mohr applied it to trusses. It is noted that p', D', r' are not the deformations produced by the loads P.

Eq. 6 may be restated in the form

$$\sum P'\,p - \sum S'\,D + \sum R'\,r = 0 \qquad (7)$$

in which P', S', R' are a statically possible set of values of the loads, stresses, and reactions. A possible form of Eq. 7 is

$$\sum p\,\delta P - \sum D\,\delta S + \sum r\,\delta R = 0 \qquad (8)$$

in which δP, δS, δR are a statically possible set of increments of P, S, R.

Complementary Energy.—The preparations have now been made for the study of the second fundamental type of variation; namely, a variation of the state of stress, during which the statical requirements of equilibrium are the ones that remain satisfied, while the geometrical requirements of continuity are abandoned temporarily. As before, P and r are considered to remain constant, but now the deformations D are interpreted as functions of the stresses S, and values S_0 are arbitrarily chosen lower limits of the stresses. Then the complementary energy will be defined by the expression

$$U = \sum \int_{S_0}^{S} D\,dS - \sum r\,R \qquad (9)$$

An infinitesimal variation from any one of the statically possible states of stress to an adjacent statically possible state gives U an increment, a first variation, equal to

$$\delta U = \sum D\,\delta S - \sum r\,\delta R \qquad (10)$$

If the set of values of D and r happen to satisfy the geometrical requirements of continuity, then these values must satisfy Eq. 8 with $\delta P = 0$. It follows that

$$\delta U = 0 \qquad (11)$$

That is, the first variation of U vanishes; the complementary energy, U, is a minimum or a maximum, like the energy T ordinarily a minimum. The statement

$$U = \min \qquad (12)$$

is interpreted with the same reservation that was applied to Eq. 5: under abnormal circumstances the minimum may be changed into a maximum. Eq. 12, with U defined by Eq. 9, is the law of least complementary energy. This minimum is produced by a variation of the state of stress.

Analogy.—A comparison of Eqs. 1 to 5 with Eqs. 8 to 12, in order, shows that a complete analogy exists between the two laws of least energy and least

complementary energy, with a one to one correspondence of the quantities, as follows: in the shifting from one principle to the other the quantities P, S, R, T, U, p, D, r are replaced by the same quantities in the reversed order.

Castigliano's Principle as Special Case.—When Hooke's law applies and a stress-less state can be and is chosen as starting state, and the lower limits S_0 are chosen as zero, the part of the complementary energy contributed by the stresses, the first sum in Eq. 9, becomes

$$U_i = \tfrac{1}{2} \sum D\,S \qquad (13)$$

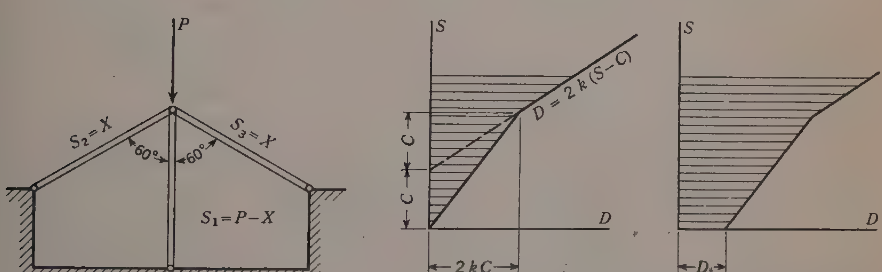
which is the same as the internal energy of stresses. If the supports have not moved, then $r = 0$. In this special yet general case the law of least complementary energy becomes

$$U_i = \min \qquad (14)$$

which is Castigliano's principle of least work, or, the principle of minimum of the internal energy by variation of the state of stress.

Application to a Simple Statically Indeterminate Structure Stressed Beyond the Proportional Limit

The simple truss in Fig. 2(*a*) will serve as an example. The stresses are total stresses, positive in compression. The state of stress is varied by varying X. The three members are assumed to be alike, with the stress-deformation diagram shown in Fig. 2(*b*). The supports are assumed not to have moved.



(*a*) STATICALLY POSSIBLE COMPRESSIVE STRESSES IN TRUSS (*b*) STRESS-DEFORMATION DIAGRAM (*c*) STRESS-DEFORMATION DIAGRAM AFTER COOLING

FIG. 2

When none of the stresses exceeds the proportional limit $2\,C$, the complementary energy is the same as the stress energy, which is

$$U = \tfrac{1}{2}\,k\,(P - X)^2 + k\,X^2 \qquad (15)$$

In Eq. 15, U must be a minimum by Castigliano's principle, which gives $X = \dfrac{P}{3}$.

When $P > 3\,C$, the stress in the vertical member will exceed the proportional limit $2\,C$. The complementary energy is determined by adding areas such as the shaded area in Fig. 2(*b*). If the proportional limit is not exceeded

in the inclined members, the total complementary energy is

$$U = k\,C^2 + k\,(P - X - C)^2 + k\,X^2 \qquad (16)$$

In Eq. 16, U becomes a minimum when

$$X = \frac{P - C}{2} = S_2 = S_3 \qquad (17a)$$

and

$$S_1 = \frac{P + C}{2} \qquad (17b)$$

Eqs. 17 apply when $3\,C < P < 5\,C$. If $P > 5\,C$, the complementary energy is

$$U = 3\,k\,C^2 + k\,(P - X - C)^2 + 2\,k\,(X - C)^2 \qquad (18)$$

which becomes a minimum when

$$X = \frac{P + C}{3} = S_2 = S_3 \qquad (19a)$$

and

$$S_1 = \frac{2\,P - C}{3} \qquad (19b)$$

Computations such as these are simplified in less simple applications by expressing the derivatives that determine the minimum, without computing the value of the complementary energy itself.

If the support of the vertical member in Fig. 2(a) settles a distance c downward, the path of the corresponding upward reaction R is $r = -\,c$, and U is increased by the amount $-\,r\,R = c\,(P - X)$. Cooling of a member merely shifts the stress-deformation diagram as indicated in Fig. 2(c), adding a term $D_t\,S$ to the complementary energy contributed by that member.

In dealing with a structure of some complexity the choice of method is usually important, because one method is likely to be less inconvenient than others that are available; but the simple structure in Fig. 2 was chosen so that the two laws of least complementary energy and least energy lend themselves equally well to it. The use of the latter law will be shown for the purpose of comparison of the two procedures.

The shape is varied by moving the top joint downward a variable distance Y. Geometrical continuity is preserved when the deformations are $D_1 = Y$ and $D_2 = D_3 = \frac{1}{2}\,Y$ and $p = Y$. The strain energy of each member is measured by an area under the lines in Fig. 2(b). It is justifiable to call this quantity energy because the stress is a definite function of the deformation in the operations of the analysis; the quantity need not be energy in every physical sense; it is energy in the analysis. If the proportional limit has been exceeded in the vertical member only, the energy in Eq. 2 becomes

$$T = \left(\frac{Y^2}{4\,k} + C\,Y - k\,C^2\right) + \frac{1}{k}\left(\frac{Y}{2}\right)^2 - P\,Y \qquad (20)$$

In Eq. 20, T is a minimum when $Y = k\,(P - C)$, which reproduces the stresses in Eqs. 17.

Buckling of Column with Hinged Ends

When a column buckles under a critical load, the movement of the loaded end, the shortening of the chord, is proportional to the square of the lateral deflections; this is a departure from linearity and from the law of superposition; but the law of superposition of deformations will hold with good approximation in the significant applications if the starting shape is chosen close to the final shape.

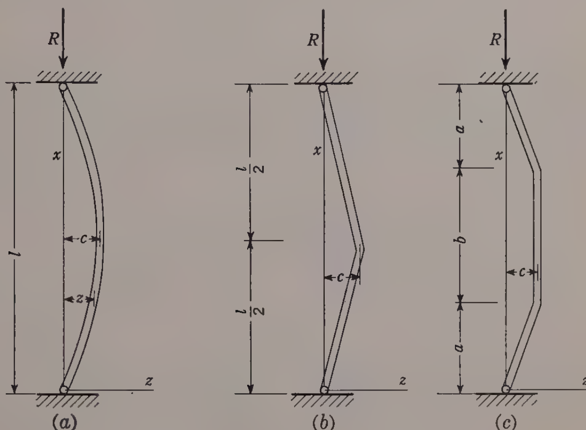


Fig. 3.—Starting Shapes of Column

Fig. 3(a) shows a plausible starting shape for a hinged-ended column, held between two fixed blocks. The blocks give the end pressure the character not of a load but of a reaction R.

Let: z = deflection in the starting shape; y = additional deflection from the starting shape to the final shape, making the total final deflection $y + z$; and $E\,I$ = modulus of elasticity times moment of inertia of the cross section.

In computing the variation of the complementary energy according to Eq. 10, S is taken as the bending moment $R\,z$ in the starting shape; this is approximately correct if the starting shape is close to the final shape. Then δS becomes $z\,\delta R$. The members are of infinitesimal length dx; therefore the summation is replaced by an integral. The deformation D becomes the relative change of slope

$$-\frac{d^2y}{dx^2}\,dx = \left(\frac{R\,z}{E\,I} + \frac{d^2z}{dx^2}\right)dx \qquad (21)$$

The displacement r is zero since the end blocks do not move. Thus one finds

$$\delta U = \int_0^l \left(\frac{R\,z}{E\,I} + \frac{d^2z}{dx^2}\right) dx\;(z\,\delta R) = 0 \qquad (22)$$

which gives the approximate formula

$$R = -\frac{\displaystyle\int_0^l z\,\frac{d^2z}{dx^2}\,dx}{\displaystyle\int_0^l \frac{z^2\,dx}{E\,I}} \qquad (23)$$

By choosing as a starting shape the parabola

$$z = 4\,c\,l^{-2}\,(l\,x - x^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

and assuming $E\,I$ constant, one finds by easy integrations

$$R = \frac{10\,E\,I}{l^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

which exceeds the correct value $\frac{\pi^2 E\,I}{l^2}$ only by 1.3%.

The same results are obtained by a variant of the energy method to which reference will be made shortly. First, however, the direct use of the law of least energy will be shown. The end pressure is now interpreted not as a reaction but as a load P. If the deflections are z, Eq. 2 takes the form

$$T = \tfrac{1}{2}\int_0^l E\,I\left(\frac{d^2z}{dx^2}\right)^2 dx - P\cdot\tfrac{1}{2}\int_0^l\left(\frac{dz}{dx}\right)^2 dx \quad \ldots\ldots\ldots (26)$$

The variation δT vanishes only when P has the critical value that makes $T = 0$. This gives

$$P = \frac{\int_0^l E\,I\left(\frac{d^2z}{dx^2}\right)^2 dx}{\int_0^l\left(\frac{dz}{dx}\right)^2 dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

With z as in Eq. 24 and $E\,I$ constant, Eq. 27 gives $P = 12\,E\,I\,l^{-2}$ which is 22% too great. The method of complementary energy gives the same result when z is taken as in Fig. 3(b); the numerator in Eq. 23 then being computed as c times the change of slope at the middle of the column. With the starting shape in Fig. 3(c) the method of complementary energy gives $R = \frac{6\,E\,I}{2\,a^2 + 3\,a\,b}$, which has its smallest and therefore most plausible value, $10.67\,E\,I\,l^{-2}$, when $a = \frac{3\,l}{8}$.

The comparison just made indicates superiority of the method of complementary energy over the direct application of the method of the potential energy to columns; but, as has been indicated, a variant of the energy method, applicable to columns, is available. S. Timoshenko[14] has described and used it. This improved variant can be explained by noting that the numerator in Eq. 27 is improved by replacing $\frac{d^2z}{dx^2}$ by $-\frac{P\,z}{E\,I}$. Then the formula becomes

$$P = \frac{\int_0^l\left(\frac{dz}{dx}\right)^2 dx}{\int_0^l \frac{z^2\,dx}{E\,I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

Eq. 28 can be rewritten in the form of Eq. 23. That is, Timoshenko's variant

[14] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., 1936, p. 81.

of the energy method for columns gives the same results as the method of complementary energy.

Column with Initial Curvature

If the column that has been under discussion is slightly curved before loading, with the initial deflections z_0, and if z still denotes the total deflections in the starting shape, Eq. 22 will be changed to

$$\delta U = \int_0^l \left(\frac{R\,z}{E\,I} + \frac{d^2 z}{dx^2} - \frac{d^2 z_0}{dx^2} \right) dx\,(z\,\delta R) = 0 \qquad (29)$$

Assume that the shape before loading makes it plausible to choose z proportional to z_0. Denote by Q the critical value of R defined by Eq. 23 for $z_0 = 0$. Then Eq. 29 gives

$$R - \left(1 - \frac{z_0}{z}\right) Q = 0 \qquad (30)$$

or,

$$\frac{z}{z_0} = \frac{Q}{Q - R} \qquad (31)$$

Eq. 31 defines a magnification factor for the deflections. Such factors will be discussed again later.

Greatest Load on an Initially Curved Column Stressed Beyond the Proportional Limit

If the column in Fig. 3(*a*) is stressed beyond the proportional limit, it may be plausible to assume a law of deformations of the type,

$$\text{Change of curvature} = \frac{1}{E_r\,I}\,(M + F^{-2}\,M^3) \qquad (32)$$

in which E_r and F are functions of the end pressure R and dependent on the material and the shape of the cross section; E_r being a reduced modulus of elasticity, and F a coefficient measurable in inch-pounds. The theory developed by Engesser (from 1889), A. Considère (1891), F. S. Yasinsky (1895), and Theodor von Kármán, M. Am. Soc. C. E., (1910),[15] furnishes a method of computing E_r; for example, this method gives for a rectangular cross section

$$E_r = \frac{4\,E\,E_t}{(\sqrt{E} + \sqrt{E_t})^2} \qquad (33)$$

in which E_t is the tangent modulus of elasticity. The theory also will make it possible[16] to establish reasonable values of F. Moreover, it should be possible to determine both E_r and F by tests of short specimens under eccentric pressure.

Let z_0 denote, as before, the initial deflections before loading, and z the total starting deflections under the pressure R. With Eq. 32 accepted, Eq. 29

[15] For explanation of the basis of the theory and for references, see "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., 1936, pp. 157–159.

[16] See, for example, the paper "Strength of Steel Columns," by H. M. Westergaard and William R. Osgood, Members, Am. Soc. C. E., *Transactions*, A. S. M. E., Vol. 50, 1928, No. 17, p. 65.

changes into the following expression of the law of least complementary energy:

$$\frac{dU}{dR} = \int_0^l \left[\frac{1}{E_r I} (R z + F^{-2} R^3 z^3) + \frac{d^2 z}{dx^2} - \frac{d^2 z_0}{dx^2} \right] z \, dx = 0 \quad \text{.....(34)}$$

Let z_0 be assumed and z chosen as follows:

$$z_0 = c_0 \sin \frac{\pi x}{l} \quad \text{.....(35a)}$$

and

$$z = c \sin \frac{\pi x}{l} \quad \text{.....(35b)}$$

and denote,

$$Q = \frac{\pi^2 E_r I}{l^2} \quad \text{.....(36)}$$

Then Eq. 34 gives

$$R c + \frac{3}{4} \frac{(R c)^3}{F^2} = Q (c - c_0) \quad \text{.....(37)}$$

In the course of a test in a laboratory, as the distance between the end blocks is reduced, the maximum deflection c will increase gradually and may serve as a measure of the progress of the test. The end pressure R will be a function of c, dependent on the initial deflection c_0. It is desired to determine the maximum value of R as a function of c, for different definite values of c_0 and l.

It is noted that E_r in Eq. 36 is a function of R; therefore Q in Eqs. 36 and 37 is a function of R. With this in view Eq. 37 shows that in the special case of $c_0 = 0$, the maximum value of R occurs when $c = 0$, and is equal to Q, as it should be in accordance with the theory initiated by Engesser. With $Q = R$ and R chosen, Eq. 36 can be solved for l, thus serving as a formula for the idealized case of an initially straight column.

When the column is curved before loading (that is, when $c_0 > 0$), Q loses its significance as a maximum load and becomes merely a quantity introduced for convenience and defined by an equation. To make R a maximum the condition $\frac{dR}{dc} = 0$ is imposed. Since Q is a function of R, it follows that $\frac{dQ}{dc} = 0$. Under these conditions, by applying the operators $3 - c \frac{d}{dc}$ and $c \frac{d}{dc} - 1$ to Eq. 37, one finds

$$2 R c = Q (2 c - 3 c_0) \quad \text{.....(38a)}$$

and

$$\frac{3 (R c)^3}{2 F^2} = Q c_0 \quad \text{.....(38b)}$$

respectively. Elimination of c from Eqs. 38 gives

$$c_0 = \frac{4 F}{9 Q} \left(\frac{Q}{R} - 1 \right)^{3/2} \quad \text{.....(39)}$$

or, with Q substituted from Eq. 36,

$$c_0 = \frac{4\,F\,l^2}{9\,\pi^2\,E_r\,I}\left(\frac{\pi^2\,E_r\,I}{R\,l^2} - 1\right)^{3/2} \quad \text{(40)}$$

Eq. 40 is not a design formula, but may lend itself to interpretations of tests. If E_r and F are numerically known functions of R, the values of R and l may be chosen, and the greatest initial deflection c_0 permitting these values may be computed by Eq. 40.

If Eq. 32 is replaced by the simpler but probably less plausible formula,

$$\text{Change of curvature} = \frac{1}{E_r\,I}\,(M + F^{-1}\,M^2) \quad \text{(41)}$$

the same procedure of analysis gives, instead of Eq. 40,

$$c_0 = \frac{3\,F\,l^2}{32\,\pi\,E_r\,I}\left(\frac{\pi^2\,E_r\,I}{R\,l^2} - 1\right)^{2} \quad \text{(42)}$$

Buckling of a Thin Circular Elastic Plate with Simply Supported Edge Under a Uniform Pressure at the Edge

A solution of this problem was indicated by G. H. Bryan[17] in 1891. The differential equation for the slope in the radial direction is solved by a Bessel function of order one. When Poisson's ratio is 0.3, this exact solution gives the following value of the critical pressure per unit of length:

$$R = \frac{4.62\,E\,I}{a^2} \quad \text{(43)}$$

in which: I = moment of inertia of the cross section per unit of width; and a = radius of the plate.

The application of the method of complementary energy to this problem is not more convenient than the use of Bessel functions, but it will serve the purpose of an illustration. The procedure can be used in more complicated related problems, involving, for example, a varying thickness.

The following additional notation is used:

r = radial distance from the center of the plate.

m_r = bending moment in the radial direction per unit of width of the circumferential section on which it acts.

m_θ = bending moment in the circumferential direction per unit of width of the radial section on which it acts; both m_r and m_θ are measurable in inch-pounds per inch, or, in pounds.

z = starting deflection.

ζ = supplementary deflection, making the final deflection $z + \zeta$.

c = z at center.

$f = f(r)$ = stress function, measurable in inch-pounds.

X = constant to be determined.

μ = Poisson's ratio.

---

[17] *Proceedings*, London Mathematical Soc., Vol. 22, 1891, p. 54; also, "Theory of Elastic Stability," by S. Timoshenko, 1936, pp. 367–369.

It is permissible to assume that z, ζ, and f are functions of r only. When z is chosen so that the values of ζ will be insignificant relative to the values of z, the bending moments can be expressed by the formulas,

$$m_r = R\,z + \frac{f}{r} \qquad (44a)$$

and

$$m_\theta = R\,z + \frac{df}{dr} \qquad (44b)$$

A simple examination shows that no matter what differentiable function f is chosen, the moments in Eqs. 44 maintain equilibrium. The variation of the state of stress therefore is reduced to the variation of R and f. When the edge is simply supported, $z = 0$ and $f = 0$ at the edge.

As in the study of columns, R is interpreted as a reaction exerted by a strong support in a fixed position. The changes of curvature in the directions of m_r and m_θ, measured from the starting shape to the final shape, are $-\frac{d^2\zeta}{dr^2}$ and $-\frac{1}{r}\frac{d\zeta}{dr}$, respectively. Accordingly the law of least complementary energy takes the form

$$\delta U = \int_0^a 2\,\pi\,r\,dr\left[-\frac{d^2\zeta}{dr^2}\,\delta m_r - \frac{1}{r}\frac{d\zeta}{dr}\,\delta m_\theta\right] = 0 \qquad (45)$$

To obtain an approximate solution the following expressions are chosen:

$$z = c\left(1 - \frac{r^2}{a^2}\right) \qquad (46a)$$

and

$$f = c\,(X - R)\left(r - \frac{r^3}{a^2}\right) \qquad (46b)$$

By Eqs. 44 the corresponding moments are

$$m_r = c\,X\left(1 - \frac{r^2}{a^2}\right) \qquad (47a)$$

and

$$m_\theta = c\,X\left(1 - \frac{3\,r^2}{a^2}\right) + \frac{2\,c\,R\,r^2}{a^2} \qquad (47b)$$

The bending moments define the corresponding changes of curvature:

$$-\frac{d^2\zeta}{dr^2} = \frac{1}{E\,I}\,(m_r - \mu\,m_\theta) + \frac{d^2z}{dr^2}$$
$$= \frac{c}{E\,I}\left\{X\left[1 - \mu - (1 - 3\,\mu)\frac{r^2}{a^2}\right] - \frac{2\,\mu\,R\,r^2}{a^2}\right\} - \frac{2\,c}{a^2} \qquad (48)$$

and

$$-\frac{1}{r}\frac{d\zeta}{dr} = \frac{1}{E\,I}\,(m_\theta - \mu\,m_r) + \frac{1}{r}\frac{dz}{dr}$$
$$= \frac{c}{E\,I}\left\{X\left[1 - \mu - (3 - \mu)\frac{r^2}{a^2}\right] + \frac{2\,R\,r^2}{a^2}\right\} - \frac{2\,c}{a^2} \qquad (49)$$

By substituting from Eqs. 47, 48, and 49 in Eq. 45, and carrying out the integrations one finds

$$\frac{3\,E\,I}{\pi\,a^2\,c^2}\,\delta U = \Big[\,4\,X - (3+\mu)\,R\,\Big]\,\delta X$$

$$+\Big[-(3+\mu)\,X + 4\,R - \frac{6\,E\,I}{a^2}\Big]\,\delta R = 0 \quad\ldots\ldots\ldots\ldots(50)$$

Eq. 50 gives

$$X = \frac{3+\mu}{4}\,R \quad\ldots\ldots\ldots\ldots(51a)$$

and

$$R = \frac{24\,E\,I}{(7+\mu)\,(1-\mu)\,a^2} \quad\ldots\ldots\ldots\ldots(51b)$$

With $\mu = 0.3$, Eqs. 51 give $R = 4.70\,\frac{E\,I}{a^2}$ which is only 1.7% greater than the correct value in Eq. 43.

By the choice,

$$z = c\cos\frac{\pi\,r}{2\,a} \quad\ldots\ldots\ldots\ldots(52a)$$

and

$$f = c\,(X - R)\,r\cos\frac{\pi\,r}{2\,a} \quad\ldots\ldots\ldots\ldots(52b)$$

instead of Eqs. 46, the slightly better value $R = 4.66\,\frac{E\,I}{a^2}$ was obtained with $\mu = 0.3$.

Application to Vibration of Beams

If an elastic beam vibrates freely in one of its modes, the deflections at the time t may be stated as

$$\eta = (z + y)\cos\omega\,t \quad\ldots\ldots\ldots\ldots(53)$$

in which ω is a constant defining the period as $\frac{2\,\pi}{\omega}$, and z and y are functions of the distance x measured along the beam. The function z will be chosen so that the values of y are relatively small. Then z may be interpreted as a starting deflection and y as a supplementary deflection, both referring to the times when $\cos\omega\,t = 1$.

Let X denote the bending moment at any point due to an imagined static load defined as the product of z times the weight w per unit of length; w may be constant or a function of the distance x; X will be a function of x and will be measurable in lb-in.[2] When $\cos\omega\,t = 1$, the inertia force per unit of length is $\frac{\omega^2}{g}\,w\,(z + y)$, in which g is the acceleration due to gravity; $\frac{w}{g}$ is the mass per unit of length. When the contributions from y can be ignored, the bending moment due to the inertia forces becomes

$$M = \frac{\omega^2}{g}\,X \quad\ldots\ldots\ldots\ldots(54)$$

The use of the inertia forces converts the problem into one of statics. The quantity $\frac{\omega^2}{g}$ now may be interpreted as an adjustable reaction, playing the same rôle as R in the application to columns. Then

$$\delta M = X\,\delta\left(\frac{\omega^2}{g}\right) \qquad (55)$$

and the variation of the complementary energy becomes

$$\delta U = \int \left(\frac{M}{E\,I} + \frac{d^2 z}{dx^2}\right) dx\,\delta M = \delta\left(\frac{\omega^2}{g}\right) \int \left(\frac{\omega^2 X}{E\,I\,g} + \frac{d^2 z}{dx^2}\right) X\,dx = 0 \qquad (56)$$

Eq. 56 gives

$$\omega^2 = -\frac{\int X \frac{d^2 z}{dx^2}\,dx}{\int \frac{X^2\,dx}{E\,I\,g}} \qquad (57)$$

from which the period may be computed as $\frac{2\pi}{\omega}$.

It is of interest to compare Eq. 57 with the next equation, which represents Rayleigh's[18] method. The left side of the equation is twice the kinetic energy when $\cos \omega t = 0$; the right side is twice the internal potential energy when $\cos \omega t = 1$; and either side is twice the total mechanical energy:

$$\omega^2 \int \frac{w}{g} z^2\,dx = \int E\,I \left(\frac{d^2 z}{dx^2}\right)^2 dx \qquad (58)$$

When the deflection z is stated approximately, Eq. 58 gives an approximate value of ω^2. The example that follows will show that Eq. 58 requires a more accurate statement of z than does Eq. 57 for the same degree of approximation in the value of ω.

A cantilever of length l, with a fixed support at $x = 0$, and with w and $E\,I$ constant, is chosen as example. An exact solution[19] of the proper differential equation gives an expression for the extreme deflections $z + y$ in terms of two hyperbolic and two trigonometric functions. Thereby the value of ω in the first mode of vibration is found to be

$$\omega = 3.516\,\omega_0 \qquad (59)$$

in which ω_0 denotes the quantity

$$\omega_0 = \sqrt{\frac{E\,I\,g}{w\,l^4}} \qquad (60)$$

In an approximate solution by the method of complementary energy the starting curve may be taken as a parabola,

$$z = \frac{c\,x^2}{l^2} \qquad (61)$$

[18] "Theory of Sound," by Lord Rayleigh, 1877, 2d Ed., MacMillan & Co., Vol. 1, 1894, pp. 109 and 257; see also "Vibration Problems in Engineering," by S. Timoshenko, Van Nostrand Co., 1928, pp. 55–60.

[19] See "Vibration Problems in Engineering," by S. Timoshenko, Van Nostrand Co., 1928, p. 234.

Then

$$X = -\frac{w\,l^2\,c}{12}\left[\left(\frac{x}{l}\right)^4 - \frac{4\,x}{l} + 3\right] \qquad (62)$$

When these functions are substituted in Eq. 57, one finds

$$\omega = \omega_0 \sqrt{\frac{162}{13}} = 3.530\,\omega_0 \qquad (63)$$

which differs only by 0.4% from the correct value in Eq. 59.

Exactly the same result can be obtained by Rayleigh's method, with Eq. 58, but the function z must be chosen much closer to the true deflections; the function z equal to the deflection under a uniform load will serve the purpose.[20] If z were taken from Eq. 61, Eq. 58 would give $\omega = \omega_0\,\sqrt{20} = 4.47\,\omega_0$, which is 27% too great.

If a straight line, $z = \frac{c\,x}{l}$, is chosen as starting shape instead of the parabola in Eq. 61, the method of complementary energy gives $\omega = \omega_0 \sqrt{\frac{140}{11}} = 3.57\,\omega_0$. Rayleigh's method gives the same result when z is chosen as the deflection due to a concentrated load at the free end.[21]

Stabilizing Loads

When a load producing buckling is reversed, it becomes a stabilizing load; but it can still be studied by the theory of buckling. The general theorems which are needed in subsequent applications of the method of complementary energy are derived most conveniently by considering the energy as a function of the shape. The derivations will be shown in brief form.[22]

It is assumed that any shape of the structure that is geometrically possible, when continuity is maintained, can be defined by assigning a definite set of values to a set of parameters $u_0, u_1, u_2, \cdots u_n, \cdots$. It is assumed that these parameters define any internal deformation by a linear equation of the form

$$D = D_0\,u_0 + D_1\,u_1 + D_2\,u_2 + \cdots + D_n\,U_n + \cdots = \sum_{0,1,2,\ldots}^{n} D_n\,u_n \qquad (64)$$

Besides, it is assumed that each member obeys Hooke's law. Then each stress will be a linear function of the parameters, and the strain energy will be a quadratic function of the parameters.

Two loads, P and Q, are considered. It is assumed that the path p of P can be stated adequately as a linear function of the parameters,

$$p = \sum_{1,2,\ldots}^{n} p_n\,u_n \qquad (65)$$

but that a quadratic function of the parameters is required for an adequate

[20] "Vibration Problems in Engineering," by S. Timoshenko, Van Nostrand Co., 1928, p. 59.

[21] *Loc. cit.*, p. 58.

[22] A fuller account of the theory represented by Eqs. 64 to 73 and 78 to 80 and explanations of the special terminology are given in the paper, "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. 85 (1922), p. 576, especially pp. 604–637.

statement of the path q of Q. It will be assumed that a special choice of parameters has made it possible to state the path q as

$$q = -q_0 u_0 - \tfrac{1}{2} \sum_{1,2,\dots}^{n} q_n u_n^2 \quad \dots\dots\dots (66)$$

and the strain energy as

$$V = \tfrac{1}{2} V_0 u_0^2 + \tfrac{1}{2} \sum_{1,2,\dots}^{n} Q_n q_n u_n^2 \quad \dots\dots\dots (67)$$

with all the coefficients q_0, q_n, V_0, and Q_n positive and all of the values Q_n different from one another. That this combination of relations is not only possible, but typical of buckling and stabilizing loads, will be brought out by the discussion that follows.

With Eqs. 65 to 67 accepted, the principle of minimum of the potential energy by variation of the shape takes the form,

$$T = V - Q\,q - P\,p = \tfrac{1}{2} V_0 u_0^2 + \tfrac{1}{2} \sum_{1,2,\dots}^{n} Q_n q_n u_n^2$$

$$+ Q \left(q_0 u_0 + \tfrac{1}{2} \sum_{1,2,\dots}^{n} q_n u_n^2 \right) - P \sum_{1,2,\dots}^{n} p_n u_n \quad \dots\dots\dots (68)$$

$$\delta T = 0, \quad \text{or,} \quad \frac{\partial T}{\partial u_n} = 0 \qquad \text{for } n = 0, 1, 2, \cdots \quad \dots\dots\dots (69)$$

Eqs. 68 and 69 will be applied to investigate a series of four actions, in which u_n will be denoted successively u_n, $\bar{u}_n$, $\bar{\bar{u}}_n$, and u_n. Table 1 will serve as a summary of the results.

TABLE 1.—SUMMARY OF THEORY OF BUCKLING AND STABILIZING LOADS

Action	Definition	Value of parameter u_n for $n = 1, 2, 3, \cdots$	Equation No.
Astatic	$P = 0,\ Q = -Q_n$	u_n = any value, others $= 0$	71
Orthostatic	$P \neq 0,\ Q = 0$	$\bar{u}_n = \frac{P\,p_n}{Q_n\,q_n} = \frac{Q}{Q_n}\bar{\bar{u}}_n$	73, 77
Relaxed	$P \neq 0,\ Q > 0$, V reduced	$\bar{\bar{u}}_n = \frac{P\,p_n}{Q\,q_n} = \frac{Q_n}{Q}\bar{u}_n$	77, 73
Heterostatic	$P \neq 0,\ Q \neq 0$, V restored	$u_n = \frac{Q_n}{Q_n + Q}\bar{u}_n = \frac{Q}{Q_n + Q}\bar{\bar{u}}_n$	80, 81

Astatic Action.—The first action is defined by

$$P = 0, \qquad Q = -Q_n \quad \dots\dots\dots (70)$$

Inspection of Eq. 68 shows that $\delta T = 0$ when

$$u_0 = \frac{Q_n q_0}{V_0}, \quad u_n = \text{any value, all other parameters } u_m = 0 \quad \dots\dots (71)$$

Since the parameter u_n may pass through a continuous range of values, defining a continuous range of shapes of the structure, the equilibrium is called neutral, and the action is described as buckling under the critical load $-Q = Q_n$. Since all other parameters remain zero except u_0, this buckling is called pure buckling or astatic action. The load $-Q$ is called an astatic load; the parameters $u_1, u_2, \cdots u_n, \cdots$ are called astatic parameters; and u_0 is called the orthostatic parameter of the astatic action. If T is a maximum by variation of some of the parameters, the equilibrium is unstable, though neutral by variation of u_n. Further examination[23] of Eq. 68 shows that if the quadratic expressions for V and q are written with mixed terms included, such as $V_{mn}\, u_m\, u_n$ and $q_{mn}\, u_m\, u_n$, knowledge of the existence of two astatic actions, one with $u_m =$ any value, $u_n = 0$ and the other with $u_m = 0$, $u_n =$ any value, at two different values of the astatic load, leads to the conclusion that $V_{mn} = q_{mn} = 0$. It follows that if the states of pure buckling can be ascertained, q and V can be written in the relatively simple forms of Eqs. 66 and 67; and thereby the nature of the astatic parameters is also ascertained.

Orthostatic Action.—The second action to be examined is defined by

$$P \neq 0, \qquad Q = 0 \quad \text{.......... (72)}$$

Eqs. 68 and 69 give $u_0 = 0$ and the following value of u_n for $n = 1, 2, \cdots$:

$$\bar{u}_n = \frac{P\, p_n}{Q_n\, q_n} \quad \text{.......... (73)}$$

The parameters are proportional to the load P; the same applies to the deformations and stresses, which are linear functions of the parameters. No buckling is involved; therefore this is the usual action dealt with in structural mechanics. In the terminology of the theory of buckling, this action, in which the astatic load is absent, is called orthostatic action. The load P is called the orthostatic load. Eq. 73 defines the values of the astatic parameters in the orthostatic action.

Relaxed Action.—In the third action to be examined the structure is assumed to have been modified by a relaxation of stiffness. The strain energy V in Eq. 67 is reduced in the relaxed structure by removing all terms except the first, so that now

$$V = \tfrac{1}{2}\, V_0\, u_0^2 \quad \text{.......... (74)}$$

It will be assumed that

$$P \neq 0, \qquad Q > 0 \quad \text{.......... (75)}$$

When the proper terms are omitted in Eq. 68 one finds

$$u_0 = -\frac{Q\, q_0}{V_0} \quad \text{.......... (76)}$$

and the following value of u_n for $n = 1, 2, \cdots$:

$$\bar{\bar{u}}_n = \frac{P\, p_n}{Q\, q_n} \quad \text{.......... (77)}$$

[23] "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. 85 (1922), p. 576, especially pp. 611–612.

Eq. 77 defines the values of the astatic parameters in the relaxed action. The equilibrium is stable, and Q is a stabilizing load.

Heterostatic Action.—In the fourth and last action to be considered the structure is assumed to have recovered its stiffness. Both the orthostatic and the astatic loads are present, that is,

$$P \neq 0, \qquad Q \neq 0 \qquad (78)$$

This general type of action is called heterostatic action; the combined load P, Q is a heterostatic load, with the orthostatic component P and the astatic component Q. By Eqs. 68 and 69 one finds u_0 as in Eq. 76, and for $n = 1, 2, 3, \cdots$:

$$u_n = \frac{P\, p_n}{(Q_n + Q)\, q_n} \qquad (79)$$

A comparison of Eq. 79 with Eqs. 73 and 77 shows that

$$\frac{u_n}{\bar{u}_n} = \frac{Q_n}{Q_n + Q} \qquad (80)$$

and

$$\frac{u_n}{\bar{\bar{u}}_n} = \frac{Q}{Q_n + Q} \qquad (81)$$

When Q is positive, Eq. 80 defines the reduction factor of the astatic parameter u_n by transition from the orthostatic action to the heterostatic action under the influence of the stabilizing load Q; and Eq. 81 defines the reduction factor of the same parameter by transition from the relaxed action to the heterostatic action by restoration of the stiffness. When Q is negative, the factors in Eqs. 80 and 81 become magnification factors. It may be noted in passing that $\frac{Q}{Q_n}$ is the transition factor $\frac{\bar{u}_n}{\bar{\bar{u}}_n}$ by change from the relaxed to the orthostatic action.

Example.—A slender horizontal tension member bending under its own weight will serve as illustration. The ends are assumed to be hinged, and the end load central. The total horizontal tension is the stabilizing load Q. The weight, w per unit of length, is the orthostatic load. Assume that the cross section is constant, with area A and moment of inertia I. Let E = modulus of elasticity, l = length, x = distance measured from one end. Then the nth critical value of the astatic load $-Q$ is defined by Euler's formula

$$Q_n = \frac{n^2\, \pi^2\, E\, I}{l^2} \qquad (82)$$

and the deflections maintained by it are

$$y = u_n \sin \frac{n\, \pi\, x}{l} \qquad (83)$$

in which u_n is a proper astatic parameter. As orthostatic parameter of the orthostatic action one may introduce the total shortening of the center line,

$u_0 = -\frac{Q\,l}{E\,A}$, which makes $q_0 = 1$. The parameter u_0 presents no difficulties in this problem.

In the relaxed action the member is changed into a cable, which deflects according to the law

$$-\frac{d^2y}{dx^2} = \frac{w}{Q} \qquad (84)$$

Eq. 84 may be rewritten so that the right side expresses a sum of corresponding effects in astatic actions, as follows:[24]

$$-\frac{d^2y}{dx^2} = \frac{4\,w}{\pi\,Q} \sum_{1,3,\ldots}^{n} \frac{1}{n} \sin\frac{n\,\pi\,x}{l} \qquad (85)$$

When the stiffness is restored, each term in the summation in Eq. 85 is multiplied by a reduction factor defined by Eqs. 81 and 82. After multiplying by $-E\,I$, one obtains in this way the bending moments in the heterostatic action,

$$M = \frac{4\,w\,l^2}{\pi^3} \sum_{1,3,\ldots}^{n} \frac{Q_1}{n\,(Q_n + Q)} \sin\frac{n\,\pi\,x}{l} \qquad (86)$$

The series in Eq. 86 converges and defines the stresses.

Eq. 86 may also be derived by way of the orthostatic action. When $Q = 0$,

$$\frac{d^4y}{dx^4} = \frac{w}{E\,I} = \frac{4\,w}{\pi\,E\,I} \sum_{1,3,\ldots}^{n} \frac{1}{n} \sin\frac{n\,\pi\,x}{l} \qquad (87)$$

After two integrations one finds the bending moment in the orthostatic action

$$\bar{M} = \frac{4\,w\,l^2}{\pi^3} \sum_{1,3,\ldots}^{n} \frac{1}{n^3} \sin\frac{n\,\pi\,x}{l} \qquad (88)$$

When Q is introduced, and the reduction factors are applied according to Eq. 80, Eq. 86 is reproduced.

The general theory may be applied to the problem in other ways. For example, the theory will furnish the series needed first, such as those in Eqs. 85 and 87; this is important for the more complicated problems in which the corresponding series are not simple trigonometric series. The computations may be made as follows: The contribution of the deflections $y = u_n \sin\frac{n\,\pi\,x}{l}$ to the path of the distributed load w (which takes the place of P in Eq. 68) is

$$p = p_n\,u_n = \int_0^l y\,dx = \frac{2\,l\,u_n}{n\,\pi} \qquad (89)$$

if n is uneven, and zero if n is even; and the corresponding contribution to the path of Q is

$$q = -\tfrac{1}{2}\,q_n\,u_n^2 = -\tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx = -\frac{n^2\,\pi^2\,u_n^2}{4\,l} \qquad (90)$$

[24] See "A Short Table of Integrals," by B. O. Peirce, 2d Ed., Ginn & Co., 1910, formula 808.

The last two equations define p_n and q_n directly, and thereafter $\bar{u}_n$ through Eqs. 73 and 82. Thus one finds the deflections in the orthostatic action

$$y = \sum_{1,3,\ldots}^{n} \bar{u}_n \sin \frac{n \pi x}{l} = \frac{4 w l^4}{\pi^5 E I} \sum_{1,3,\ldots}^{n} n^{-5} \sin \frac{n \pi x}{l} \ldots\ldots\ldots (91)$$

The deflections in the heterostatic action are obtained by applying the reduction factor in Eq. 80 to each term in Eq. 91. Eq. 86 is reproduced easily by this method. Another route to the same result is to determine $\bar{u}_n$ by Eq. 77, and thereby the deflections in the relaxed action; from these one finds the deflections in the heterostatic action by applying the reduction factor in Eq. 81 to each term in the series.

The method of complementary energy is applicable to problems of stabilizing loads through its applicability to astatic actions.

Stiffening of Suspension Bridges

That a suspension bridge is stiffened by its own weight was brought out in an analysis published by Joseph Melan[25] in 1888. Leon S. Moisseiff, M. Am. Soc. C. E., developed the theory further for the design of the Manhattan Bridge (1909); and F. E. Turneaure,[26] Hon. M. Am. Soc. C. E., investigated this application and contributed to, and presented, the theory. Since then this principle of stiffening has become well appreciated in America; it has been utilized in the design of the great bridges, and has been discussed in a notable American technical literature on the subject.[25, 26, 27] The dead load has the character of a stabilizing load.

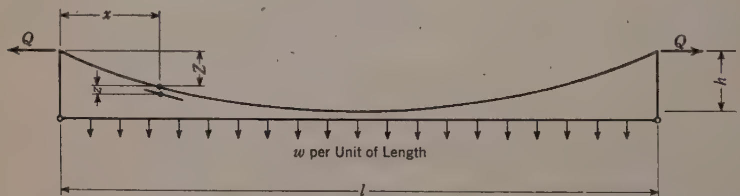


Fig. 4.—Main Span of Suspension Bridge

It is not intended here to present a full analysis of the stiffening of suspension bridges. It is desired only to show the applicability of the method of complementary energy to this problem through the theory of stabilizing loads and buckling.[28] The analysis therefore is limited to a simplified case, which is suggested in Fig. 4. The following actions are left out of account, as matters

[25] "Theorie der eisernen Bogenbrücken und Hängebrücken," by Joseph Melan, *Handbuch der Ingenieurwissenschaften*, Vol. 2, Subvolume 4, Leipzig, 1888, pp. 1–144, especially p. 38 (English translation of 3d Ed. (1906) entitled "Theory of Arches and Suspension Bridges," by D. B. Steinman, M. Am. Soc. C. E., The M. C. Clark Publishing Co., 1913).

[26] "Modern Framed Structures," by J. B. Johnson, C. W. Bryan, and F. E. Turneaure, John Wiley & Sons, Inc., Vol. 2, 9th Ed., 1911, pp. 276–318.

[27] "A Practical Treatise on Suspension Bridges," by D. B. Steinman, John Wiley & Sons Co., Inc., 1929, pp. 246–282; Leon S. Moisseiff, *Journal of the Franklin Institute*, October, 1925. *Transactions*, Am. Soc. C. E., Vol. 91 (1927), J. A. L. Waddell, pp. 884–910, especially pp. 893–895; Vol. 94 (1930), S. Timoshenko, pp. 377–391; Vol. 97 (1933), O. H. Ammann, pp. 1–65, especially pp. 39–44; Vol. 100 (1935), D. B. Steinman, pp. 1133–1170; and Vol. 104 (1939), Shortridge Hardesty and Harold E. Wessman, pp. 579–608.

[28] The application of the theory of buckling of "the suspension bridge upside-down" to suspension bridges was suggested by H. M. Westergaard in a discussion, *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 1021.

that can either be incorporated in a more extensive analysis by the same principles or be referred to independent supplementary analyses: elongations of the cable beyond the initial stretching; movements of the points of the cable at the tops of the towers; and elongations of the suspenders beyond the initial stretching. The suspenders are assumed to be spaced closely. The initial curve of the cable is assumed to be a parabola with a sag that is fairly small compared with the span. The following notation is used:

x = horizontal coordinate.

Z = initial vertical coordinate, positive downward, of any point of the cable.

z = addition to Z by the change from the initial shape to the starting shape; with the restrictions imposed, z is also the starting deflection of the stiffening truss.

y = addition to z by the change from the starting shape to the final shape.

Y = total deflection, addition to Z, in a relaxed action in which neither the cable nor the truss contributes stiffness against bending.

η = total deflection, addition to Z, in the heterostatic action.

l = span.

$h = Z_{max}$.

Q = horizontal component of the total tension in the cable; Q is interpreted as a measure of the stabilizing load.

Q_n = critical value of $-Q$ in an unstable astatic action.

w = uniformly distributed vertical load per unit of length; interpreted in this analysis as a reaction adjusting itself to Q as if it were a hydrostatic pressure.

P = orthostatic load.

K = force defined by Eq. 95.

M = sum of the bending moments in the stiffening truss and the cable at any value of x.

$E\,I$ = measure of the combined stiffness of the stiffening truss and the cable against bending.

k = ratio defined by Eq. 117.

The initial parabolic curve of the cable has the equation

$$Z = 4\,h\,l^{-2}\,(l\,x - x^2) \qquad (92)$$

If the shape is changed by infinitesimal increments δZ, the potential energy of the uniformly distributed load w will be changed by the amount $-\,w \int_0^l \delta Z\,dx$. This amount is infinitesimal of second order because it is the total change of energy if the cable is unstiffened, and the parabola is the curve of equilibrium of the unstiffened cable. Therefore, when small quantities of second order, which will be without importance in the subsequent computations, are ignored, and under the simplifying restrictions that were imposed, and as long as the deflections z, y, Y, and η remain small, it can be asserted that

$$\int_0^l z\,dx = \int_0^l y\,dx = \int_0^l Y\,dx = \int_0^l \eta\,dx = 0 \qquad (93)$$

The first task is to investigate the astatic actions at critical negative values of the loads in Fig. 4. Under the loads in Fig. 4, when the deflections are z, the combined bending moment in the cable and the truss is

$$M = \tfrac{1}{2} w (l x - x^2) - 4 Q h l^{-2} (l x - x^2) - Q z \quad\ldots\ldots\ldots(94)$$

It is expedient to introduce a force K defined by the equation

$$w = 8 (Q + K) h l^{-2} \quad\ldots\ldots\ldots(95)$$

Then

$$M = - Q z + 4 K h l^{-2} (l x - x^2) \quad\ldots\ldots\ldots(96)$$

and the increment of M by variation of the state of stress is

$$\delta M = - z \,\delta Q + 4 h l^{-2} (l x - x^2) \,\delta K \quad\ldots\ldots\ldots(97)$$

When z is chosen so that the supplementary deflections y are small, the relation of deformations to stresses

$$- \frac{d^2y}{dx^2} = \frac{M}{E I} + \frac{d^2z}{dx^2} \quad\ldots\ldots\ldots(98)$$

may be used with M as in Eq. 96.

The variation of the complementary energy may now be written:

$$\delta U = \int_0^l \left(\frac{M}{E I} + \frac{d^2z}{dx^2} \right) dx \,\delta M$$

$$= \frac{1}{E I} \int_0^l \left[- Q z + 4 K h l^{-2} (l x - x^2) + E I \frac{d^2z}{dx^2} \right] \left[- z \,\delta Q + 4 h l^{-2} (l x - x^2) \,\delta K \right] dx = 0 \quad\ldots\ldots(99)$$

This equation is converted into the following two, in which all integrations are from 0 to l:

$$E I \frac{\partial U}{\partial Q} = Q \int z^2 \, dx - 4 K h l^{-2} \int (l x - x^2) z \, dx - E I \int z \frac{d^2z}{dx^2} \, dx = 0 \quad\ldots\ldots\ldots(100)$$

$$E I \frac{\partial U}{\partial K} = - 4 Q h l^{-2} \int (l x - x^2) z \, dx + 16 K h^2 l^{-4} \int (l x - x^2)^2 \, dx + 4 E I h l^{-2} \int (l x - x^2) \frac{d^2z}{dx^2} \, dx = 0 \quad\ldots\ldots\ldots(101)$$

Eqs. 100 and 101 are satisfied by the trivial solution $z = 0$, $K = 0$, $Q =$ any value. Non-trivial solutions exist, however. To find them, the following forms of z are suitable:

$$z = z_n \sin \frac{n \pi x}{l} \qquad \text{for } n = 2, 4, 6, \cdots \quad\ldots\ldots\ldots(102)$$

and

$$z = z_n \left(\sin \frac{n \pi x}{l} - \frac{1}{n} \sin \frac{\pi x}{l} \right) \qquad \text{for } n = 3, 5, 7, \cdots \ldots\ldots (103)$$

Both satisfy the requirement in Eqs. 93.

With z as in Eq. 102 the solution is

$$K = 0, \qquad -Q = Q_n = n^2 \pi^2 \frac{E I}{l^2}, \qquad \text{for } n = 2, 4, \cdots \ldots\ldots (104)$$

This result could have been found more easily, without invoking the method of complementary energy, by noting that Eq. 102 is the solution of Eq. 98 with $y = 0$, and with K and Q as in Eq. 104.

It is when the diagram of deflections is symmetrical, as by Eq. 103, that the method of complementary energy shows merit; because Eq. 103 will be close, although not identical, to an exact solution of Eq. 98 with $y = 0$. With z as in Eq. 103, Eqs. 100 and 101 become, after multiplication by $\frac{2}{l}$:

$$(1 + n^{-2})\, z_n^2\, Q + 32\, (n^{-1} - n^{-3})\, \pi^{-3}\, h\, z_n\, K$$
$$+ (n^2 + n^{-2})\, \pi^2\, E\, I\, z_n^2\, l^{-2} = 0 \ldots\ldots\ldots\ldots\ldots (105)$$

and

$$32\, (n^{-1} - n^{-3})\, \pi^{-3}\, h\, z_n\, Q + \frac{16}{15}\, h^2\, K = 0 \ldots\ldots\ldots\ldots (106)$$

Eq. 106 shows that K is proportional to Q but also proportional to the small ratio $\frac{z_n}{h}$; therefore K is a small supplementary force, although its influence may not be small. Elimination of K gives

$$[1 + n^{-2} - 960\, (n^{-1} - n^{-3})^2\, \pi^{-6}]\, Q = -\,(n^2 + n^{-2})\, \pi^2\, E\, I\, l^{-2} \ldots (107)$$

Since π^6 is very close to 960, and n^{-6} will be insignificant, the left side of Eq. 107 can be restated as $(1 + 2\, n^{-4})\, Q$. A further simplification, representing no significant loss of accuracy, reduces the solution to the form

$$-Q = Q_n = (n^2 - n^{-2})\, \pi^2 \frac{E I}{l^2}, \qquad \text{for } n = 3, 5, 7, \cdots \ldots\ldots (108)$$

Exact values of Q_n for $n = 3, 5, 7, \cdots$ can be found by solving Eq. 98 with $y = 0$ and M as in Eq. 94. It is found that these values are defined by the equations

$$\tan \alpha = \alpha + \frac{\alpha^3}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (109a)$$

and

$$Q_n = 4\, \alpha^2 \frac{E I}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (109b)$$

Furthermore, it is found that Eq. 108 gives a very close approximation to these exact values.

This completes the study of the astatic actions, but the goal is the heterostatic action. This goal may be reached either by the route of the orthostatic action or by the route of the relaxed action, in which all stiffness identified with elasticity is removed. The latter route is chosen.

The stabilizing load Q consists of the two horizontal forces Q in Fig. 4 and the distributed load w which is proportional to Q. An added vertical load is an orthostatic load if it requires no change of Q in the relaxed action. Let M_0 denote the bending moments that such a load would produce in a simple beam replacing the bridge. Simple considerations of statics show that in the relaxed action this load produces the deflections

$$Y = \frac{M_0}{Q} \qquad (110)$$

By referring to Eqs. 93 one obtains the condition

$$\int_0^l M_0 \, dx = 0 \qquad (111)$$

The loads in Fig. 5 are ascertained as two examples of orthostatic loads by noting that the moments which they would produce in a simple beam satisfy

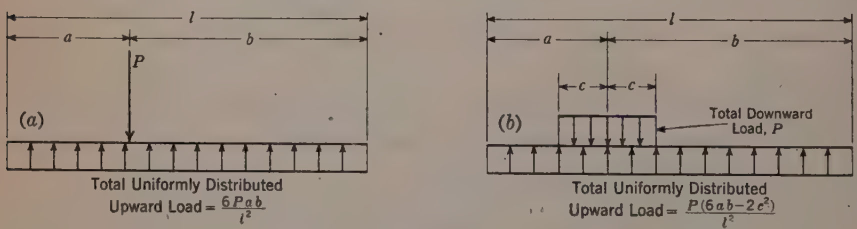


FIG. 5.—TWO EXAMPLES OF ORTHOSTATIC LOADS ON SUSPENSION BRIDGE

Eq. 111. It is observed further that any vertical load can be resolved into two component parts: one, an orthostatic load with moments satisfying Eq. 111; and the other a uniformly distributed load contributing to the astatic load Q. Therefore the function Y can be determined readily for any vertical load.

The next step is to express Y as a sum of functions representing deflections in astatic actions. The functions in Eqs. 102 and 103 are sufficiently close to the exact functions to serve this purpose. A suitable procedure is to expand the function Y into a Fourier series of the form

$$Y = \sum_{1, 2, 3, \ldots}^{n} Y_n \sin \frac{n \pi x}{l} \qquad (112)$$

with the coefficients

$$Y_n = \frac{2}{l} \int_0^l Y \sin \frac{n \pi x}{l} \, dx \qquad (113)$$

It may be convenient instead to begin by expressing the orthostatic load by a Fourier series, and then integrating twice to obtain M_0. In either case the series in Eq. 112 is obtained by a straight-forward and well-established process.

It is assured in advance that $\int_0^l Y\, dx = 0$. It follows that

$$Y_1 = - \sum_{3,\,5,\,\ldots}^{n} n^{-1}\, Y_n \quad \ldots\ldots (114)$$

Then Eq. 112 may be rewritten in the desired form

$$Y = \sum_{2,\,4,\,\ldots}^{n} Y_n \sin\frac{n\,\pi\,x}{l} + \sum_{3,\,5,\,\ldots}^{n} Y_n \left(\sin\frac{n\,\pi\,x}{l} - n^{-1}\sin\frac{\pi\,x}{l}\right) \quad \ldots (115)$$

The transition from the relaxed to the heterostatic action is made by applying the reduction factor in Eq. 81 to each term in Eq. 115, with Q_n defined by Eqs. 104 and 108. This reduction produces the deflections in the heterostatic action,

$$\eta = \sum_{2,\,4,\,\ldots}^{n} \frac{Q}{Q_n + Q}\, Y_n \sin\frac{n\,\pi\,x}{l}$$

$$+ \sum_{3,\,5,\,\ldots}^{n} \frac{Q}{Q_n + Q}\, Y_n \left(\sin\frac{n\,\pi\,x}{l} - \frac{1}{n}\sin\frac{\pi\,x}{l}\right) \quad \ldots\ldots (116)$$

By introducing the notation

$$k = \frac{Q\,l^2}{\pi^2\,E\,I} \quad \ldots\ldots (117)$$

and referring to Eqs. 104 and 108, which define the critical values Q_n of $-Q$, the final deflection, η in Eq. 116, can be restated in either of the following two forms:

$$\eta = -\sin\frac{\pi\,x}{l} \sum_{3,\,5,\,\ldots}^{n} \frac{k}{n(n^2 - n^{-2} + k)}\, Y_n$$

$$+ \sum_{2,\,4,\,\ldots}^{n} \frac{k}{n^2 + k}\, Y_n \sin\frac{n\,\pi\,x}{l} + \sum_{3,\,5,\,\ldots}^{n} \frac{k}{n^2 - n^{-2} + k}\, Y_n \sin\frac{n\,\pi\,x}{l} \quad \ldots (118)$$

or,

$$\eta = Y + \sin\frac{\pi\,x}{l} \sum_{3,\,5,\,\ldots}^{n} \frac{n^2 - n^{-2}}{n(n^2 - n^{-2} + k)}\, Y_n$$

$$- \sum_{2,\,4,\,\ldots}^{n} \frac{n^2}{n^2 + k}\, Y_n \sin\frac{n\,\pi\,x}{l} - \sum_{3,\,5,\,\ldots}^{n} \frac{n^2 - n^{-2}}{n^2 - n^{-2} + k}\, Y_n \sin\frac{n\,\pi\,x}{l} \quad \ldots (119)$$

After a certain value of n the series in Eq. 118 converge more rapidly than those in Eq. 119. For numerical computation the choice between the two formulas depends on the values of k and Y_n and on the accuracy that is desired.

Influence of Stabilizing Loads on Vibrations

A stabilizing load stiffens a structure. Therefore it will tend to reduce the periods of free vibration.

A mode of free vibration is characterized by the condition that all the significant deformations are proportional at each instant to a single parameter v which varies with the time t according to the law $v = u \cos \omega_n t$; u may have any value, is independent of t, and is a parameter corresponding to v but defining the extreme deformations.

It happens in some significant cases that the parameter u can be taken as any one of the astatic parameters u_n in Eq. 68. Then, by referring to Eq. 68, it will be seen that the part of the potential energy that varies during the vibration is $\frac{1}{2}(Q_n + Q)\, u_n^2 \cos^2 \omega_n t$. Since the velocities are proportional to $\omega_n u_n \sin \omega_n t$, the kinetic energy can be written as $\frac{1}{2} K_n \omega_n^2 u_n^2 \sin^2 \omega_n t$. Constancy of the combined potential and kinetic energy requires that

$$K_n \omega_n^2 = Q_n + Q \qquad (120)$$

If the stabilizing load Q is removed without changing the masses, ω_n and the corresponding period $t_n = \dfrac{2\pi}{\omega_n}$ will assume different values ω_{0n} and t_{0n}. Eq. 120 shows that the relations of the values are

$$\frac{t_n^2}{t_{0n}^2} = \frac{\omega_{0n}^2}{\omega_n^2} = \frac{Q_n}{Q_n + Q} \qquad (121)$$

That is, in the transition from a free vibration without the stabilizing load to one with the stabilizing load a reduction factor is applied to the square of the period equal to the reduction factor in Eq. 80, representing the transition from the orthostatic action to the heterostatic action under the stabilizing load Q.

Free Vibrations of a Suspension Bridge

While the astatic actions of a suspension bridge may be visualized by imagining gravity reversed or the bridge turned upside down, correspondingly, the free vibrations without stabilizing loads may be visualized by imagining gravity removed temporarily or the bridge merely keeled over so that cable and stiffening truss are placed in a horizontal plane.

The discussion will be limited again to the simplified case that has been under consideration. Furthermore, only vibrations in the plane of the cable and stiffening truss will be considered; and energies due to components of velocity in the direction of the span will be assumed to be relatively so insignificant that they can be ignored. Then the modes of vibration will be represented adequately by Eqs. 102 and 103. When the procedure leading to Eq. 57 is applied to Eq. 103, one finds that in the modes of $n = 3, 5, 7, \cdots$ the bending moments due to the load $w z$ are represented not exactly but with good approximation by the formula

$$X = \frac{w u_n l^2}{n^2 \pi^2} \left(\sin \frac{n \pi x}{l} - \frac{1}{n} \sin \frac{\pi x}{l} \right) \qquad (122)$$

By referring to Eq. 57, one finds, for $n = 3, 5, 7, \cdots$,

$$\omega_{0n}^2 = - \frac{\displaystyle\int X \frac{d^2z}{dx^2}\,dx}{\displaystyle\int \frac{X^2\,dx}{E\,I\,g}}$$

$$= \frac{n^2\,\pi^4\,E\,I\,g \displaystyle\int_0^l \left(\sin\frac{n\,\pi\,x}{l} - \frac{1}{n}\sin\frac{\pi\,x}{l}\right)\left(n^2\sin\frac{n\,\pi\,x}{l} - \frac{1}{n}\sin\frac{\pi\,x}{l}\right)dx}{w\,l^4 \displaystyle\int_0^l \left(\sin\frac{n\,\pi\,x}{l} - \frac{1}{n}\sin\frac{\pi\,x}{l}\right)^2 dx}$$

$$= \frac{(n^6 + n^2)\,\pi^4\,E\,I\,g}{(n^2+1)\,w\,l^4} \qquad \text{(123)}$$

or, by substitution from Eq. 108 and with a further permissible approximation,

$$\omega_{0n}^2 = \frac{n^2\,\pi^2\,g}{(1 + n^{-2} - 2\,n^{-4})\,w\,l^2}\,Q_n \qquad \text{for } n = 3, 5, 7, \cdots \qquad \text{(124)}$$

A similar but simpler computation based on Eq. 102 gives

$$\omega_{0n}^2 = \frac{n^2\,\pi^2\,g}{w\,l^2}\,Q_n \qquad \text{for } n = 2, 4, 6, \cdots \qquad \text{(125)}$$

When the suspension bridge is again placed in its natural position and gravity contributes stiffness, the values of ω_{0n}^2 in Eqs. 124 and 125 will be increased to ω_n^2 in accordance with Eq. 121 by replacing the factors Q_n by $Q_n + Q$. Thereafter the periods of free vibration are obtained as $t_n = \frac{2\,\pi}{\omega_n}$. With the critical loads Q_n substituted from Eqs. 104 and 108, and the ratio k introduced from Eq. 117, the periods of free vibration become

$$t_n = \frac{2\,l^2}{n\,\pi}\sqrt{\frac{w}{(n^2 + k)\,E\,I\,g}} \qquad \text{for } n = 2, 4, 6, \cdots \qquad \text{(126)}$$

and

$$t_n = \frac{2\,l^2}{n\,\pi}\sqrt{\frac{(1 + n^{-2} - 2\,n^{-4})\,w}{(n^2 + n^{-2} + k)\,E\,I\,g}} \qquad \text{for } n = 3, 5, 7, \cdots \qquad \text{(127)}$$

Conclusion

It has been shown that the method of complementary energy can be applied with advantage to a variety of problems in structural statics and dynamics.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MINIATURE SYSTEM OF FIRST-ORDER ALINEMENT AND TRIANGULATION CONTROL

Discussion

BY FLOYD W. HOUGH, M. AM. SOC. C. E.

FLOYD W. HOUGH,[10] M. AM. SOC. C. E. (by letter).[10a]—Interested engineers, in discussing this subject, have emphasized the fact that a high order of accuracy on either alinement or triangulation, where short lines are involved, can be obtained only by a careful evaluation of observational errors arising from various sources. Small accidental errors are inherent in any type of direct observational measurement. It is essential that studied methods of operation be used that will of themselves eliminate or make inappreciable all tendency to constant or variable systematic errors, while, at the same time, they are reducing to a minimum the amplitude of the ever-present accidental errors.

As Major Bowie has shown, there is too often a false economy practiced by the heads of engineering organizations in endeavoring to "get by" with the cheaper type of instrument. Even when it is possible to obtain finally satisfactory results from extensive observations with an inferior instrument, an analysis of total cost will show that over a short period of time the difference between initial costs of the inferior and the superior instrument will be entirely absorbed by the more rapid progress of the work with its consequent reduction in cost of labor. It should be borne in mind, too, that with reasonable care the depreciation in value of good surveying instruments is small, their life extending often from twenty to forty years or more.

The writer agrees with the principle inferred by Mr. Heaton that there is no observational advantage gained in reducing the width of slit after a certain reduced width is reached which, in the case of the Tygart Dam work, was believed to be $\frac{1}{8}$ in. If a wider slit were used, however, the use of thin paper or ground glass to eliminate possible errors of eccentricity of the bulb and reflector would be required, as Mr. Heaton has stated.

Both Mr. Dell and Professor Stanton suggest that more detail should be

NOTE.—This paper by Floyd W. Hough, M. Am. Soc. C. E., was published in December, 1939, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1940, by Messrs. William Bowie, Earl O. Heaton, George H. Dell, Charles B. Stanton, and C. L. Garner.

[10] Care, Colombian Petroleum Co., Cucuta, Colombia.

[10a] Received by the Secretary November 12, 1940.

given as to the exact sequence followed in establishing the alinement points. The proper sequence of observations is important and the writer is pleased to elaborate on this procedure as follows:

Stations AE1 and AW2 (Fig. 2) were originally fixed as the base alinement monuments from which all other points on the main alinement were derived. (Stations ASE1 and ASW1 were likewise fixed base points for the independent spillway line.) In making the extensions later to AE and AW, the telescope was plunged to determine a preliminary setting of AE and AW approximately on the fixed line AE1–AW2, within an inch or so, by use of the shuffleboard. The theodolite was then moved, for example, to AE1 and the shuffleboard to AW2. With a light on the preliminary mark at AW, a large number of observations were then read on the shuffleboard tape at AW2, using all precautions to minimize observational errors. The mean reading determined the small offset, at AW2. Knowing this offset and the distance between the monuments, the final point at AW was then found by measuring directly on the alinement bar a computed small right-angled offset from the formerly established preliminary point. A similar procedure was used in the determination of the extension point AE.

The "master alinement bar" A13 was set, as described in Table 1 and its supporting text, with the theodolite at the base point AW2, foresighting on base point AE1, then dropping down to the shuffleboard and light at A13. (It will be noted that there was a smaller vertical angle at AW2 than at AE1, and A13 was only one half as far from AW2 as from AE1.) With A13 fixed in position, the theodolite was set up on A13, and A22 was fixed by foresighting on AW2. Likewise, with the theodolite at A13, foresighting on AE1, AE4 was set. Similar procedure was used in locating all other bars on the line. The spillway line was established by corresponding methods. It is entirely independent from the main line and approximately parallel to it.

All observed points on both lines were located by foresighting on a relatively distant control point, with the exception of one monolith, A34, which was set by plunging the telescope for a short distance with the theodolite at A33 and backsighting on a distant point.

The six transverse intersections were established by use of the shuffleboard on lines connecting triangulation stations. The instrument was set on the nearest station to the intersection and foresighted on the more distant station in each case.

The suggestion has been made that it might have been advantageous to place additional alinement bars in each monolith opposite those of the main line. There are bars on both alinement lines in monoliths 13 and 22, although not directly opposite each other. Additional bars were not placed in the other monoliths as it appeared that their value would scarcely justify their cost, although this may be open to question. It is likely that any appreciable movement in a monolith would be at right angles to the face of the dam, and both bars in the same monolith would necessarily show the same upstream or downstream movement. The monoliths are separated only by narrow expansion joints. Hence, any appreciable displacement of a monolith in azimuth would

be accompanied by a similar twisting movement in adjacent monoliths, thus throwing several bars of the main line or spillway line out of position. This would subject the movement to detection by the measurement of successive increases in the offsets of adjacent monoliths. The opposite bars would serve as checks, however, by furnishing on the original alinement a comparison of the computed distance (if the two alinement lines are not exactly parallel as in this case) with the measured distance between bars of the same monolith.

The use of transceivers, or short-wave intercommunication sets, was attempted at the start of the alinement work but was discontinued in favor of the simpler flashlight signals and the international Morse code. However, there is no reason why these or similar transceivers could not be made more dependable and become entirely satisfactory. They were used with some success on other triangulation work of the Pittsburgh District. Similar equipment is used by the U. S. Army as well as by the U. S. Forest Service. It is believed that the U. S. Coast and Geodetic Survey has experimented also with short-wave communication sets to determine their adaptability to triangulation work.

In the measurement of small earth movements in regions of seismic activity, it would seem possible that alinement lines laid down across known fault lines or zones, and extending back to stable ground on each end, would serve as valuable supplements to traverse and triangulation control. They would be adaptable particularly to terrane where the lines are somewhat short for triangulation and too rough to make accurate traverse measurements. Contiguous alinement lines across fault zones should lie at widely varying azimuths up to 90° to allow detection of movement in any direction.

Commander Garner calls attention to the fact that the theodolite used at the Tygart Dam was an American product. The instrument used for the final alinement of the long tunnels of the Colorado River Aqueduct in California was of the same type, designed by the U. S. Coast and Geodetic Survey and built by an American firm. Its accuracy is at least equivalent to, and its adaptability in the field certainly exceeds, that of the finest theodolite made abroad. Although it is true that such instruments must still be made on special order in the United States, the writer believes that appreciation of the economy in using high-grade instruments is growing so rapidly that American manufacturers will soon by justified in producing these and similar instruments in sufficient quantity to enable them to reduce their costs considerably.

Major Bowie makes the following statement: "That knowledge of the degree of stability of a dam structure and of the surrounding area is desirable admits of no doubt." It is desired here to present an actual occurrence which would seem to substantiate this contention.

The failure of the St. Francis Dam in California occurred on March 13, 1927, with its attendant great loss of life and property. During a court investigation of the disaster, the City of Los Angeles stated its belief that an earth movement was responsible for the catastrophe. The writer was asked to present evidence to the court concerning this claim. He made a thorough study of the surveys in the vicinity of the dam at the time of its construction and of those subsequent

to the failure, with specific attention to the methods used and the reliability of the resultant data.

It was discovered that certain noticeable and suspicious changes did exist in angles and position, all of a consistent trend, pointing to the probability of earth movement. However, it was also found that the triangulation done just prior to the construction of the dam was of third-order accuracy. The magnitude of the apparent movements was still within the limits of observational errors for this class of control. In view of the latter fact, it was necessary, of course, to go into court and testify that, in the writer's opinion, there was not sufficient proof to warrant a conclusion that there had been any disturbance of the earth's crust contiguous to the dam.

Had a first-order alinement and triangulation control survey been made at the time of the construction of the St. Francis Dam, comparable to that made at the Tygart Dam, based on certain fixed and recognized specifications for that class of work, it would have been possible to prove what changes, if any, had taken place. Due to the lack of adequate survey control, therefore, the city had no satisfactory evidence to support its warranted suspicions of earth movement and the answer as to whether an Act of God was a contributing or even a major cause of the disaster cannot be known. Although such a failure is happily of rare occurrence in modern engineering design, it seems obvious that the ability to study and evaluate, at any time, the conditions of the physical stability of an engineering structure and of the surrounding terrane, amply justifies the expenditure of the small percentage required of the total cost of the structure.

In closing, the writer wishes to record his appreciation for the interest shown and particularly for the contributions and suggestions made by those who have discussed the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THEORY OF ELASTIC STABILITY APPLIED TO STRUCTURAL DESIGN

Discussion

BY JOSEPH S. NEWELL, ESQ.

JOSEPH S. NEWELL,[12] ESQ. (by letter).[12a]—The points of view of the civil and aeronautical structural designer being different, there is little that a practitioner in the latter field can criticize or generalize upon in this paper. From his point of view, some of the procedures described would lead to structures too conservatively proportioned for use in aircraft, but he realizes that rigidity of structure is of greater importance in civil engineering practice than is economy of weight, and concludes that a rationalization of the "rough-and-ready" rules for the stability of structural elements formerly, and probably currently, used by bridge designers is not only justifiable but urgently needed. As such, the paper merits the consideration of practicing structural designers, and of those members of regulatory committees whose duties entail the preparation or modification of design specifications.

Minor faults, based on differences of opinion or practice rather than errors in fundamentals, can be enumerated. The last sentence in the second paragraph of the "Synopsis," for instance, leads one to believe that once a structural element has formed visible waves, a small increase in the load acting upon it will cause its failure. The aircraft designer knows this is not so. A wide, thin plate subjected to compression, if properly stiffened by members running in the direction of the load, may be badly buckled in the region between stiffeners yet carry without failure a stress on, and adjacent to, the stiffeners of five, ten, fifteen or even twenty times that which caused the central part of the plate to buckle. Eq. 1 may be said to indicate the compressive stress at which the central part of such a plate will buckle, but it does not give the stress at which failure of plate and stiffener will necessarily occur. Perhaps it would be well to emphasize that where stiffened plates are used as structural elements, a load which would cause an average stress over plate and stiffener in excess of σ_{cr} given by Eq. 1, actually causes buckling in the central part of the plate when

NOTE.—This paper by Leon S. Moisseiff and Frederick Lienhard, Members, Am. Soc. C. E., was published in January, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1940, by Louis Balog, Esq.

[12] Prof. of Aeronautical Structural Eng., Mass. Inst. of Tech., Cambridge, Mass.

[12a] Received by the Secretary January 3, 1941.

σ_{cr} is reached, and a redistribution of stress near the stiffeners for stresses exceeding σ_{cr}. For such panels, the critical stress intensity may be the yield point of the material, or the stress at which the stiffener and an effective width of sheet will fail as a simple column.

The "effective width" acting on either side of the stiffener in such a case would be found by taking one half the value of d computed from Eq. 9, for a plate of given thickness, t, and a stress in the stiffener, σ. However, since the general public is not accustomed to traveling across structures so proportioned that they wrinkle under load, the civil engineer cannot take advantage of the savings possible when members are proportioned by the effective-width procedure. He must look upon wrinkling of a plate as a "failure," and proceed accordingly. Eq. 15 enables him to do this conservatively and rationally by relating stiffener size to plate size in such a manner that both will become elastically unstable under the same stress. How much more consistent the proportioning of structural elements could be if the modern structural engineer were to take advantage of rational procedures such as this instead of abiding by the dicta of his handbooks, using ratios of $\frac{d}{t} = 30, 50, 100$, or other round numbers.

Plate girder web plates may also buckle due to shear without causing failure of the girder, if the chords and stiffeners are correctly proportioned. The waves formed in such circumstances simply indicate that the compressive principal stress induced by shear in the plate has exceeded the intensity compatible with elastic stability. However, a much greater shear can be carried by the plate, since it will resist by developing large diagonal tension stresses. These stresses will tend to pull the chords of the girder together and they may cause large bending moments in the chords, and large compressive stresses in the vertical stiffeners. Here again, wrinkling of the plate simply means a redistribution of stress in the member, and a competent designer, appreciating this, can proportion a structure to be safe whether the web is "shear-resistant," or "tension-field." It is appreciated that the equanimity of truck drivers might suffer were panels in a bridge suddenly to go from one state to the other as a heavily-loaded truck passed, but the fact that structures may be impracticable does not make them impossible and the more progressive engineers should be interested in rational methods for determining where the boundary between the practicable and impracticable actually lies.

The section on "Vertical Stiffeners of Webs of Plate Girders" provides a much more rational attack on that problem than is available to a majority of designers, and the subsequent section on "Horizontal Stiffeners of Webs of Plate Girders" provides the American engineer a means of proportioning stiffeners of a type which, because of a lack of rule-of-thumb procedures in his handbooks, he seldom employs, useful though they might be. The curves presented, and the illustrative examples embodying the use of three different materials, should prove invaluable to the structural engineer interested in the use of aluminum alloy or high-strength steel.

In closing, the writer wishes to commend the authors on their presentation

of rational means for proportioning plates so that they will be elastically stable, and on their simplification of some of the more tedious equations and methods in the form of working curves. Whether the methods they describe will be adopted immediately by their professional brethren, or whether it will remain for the colleges to present the material to future students so that it may eventually be accepted, remains to be seen. In either case, the authors have taken steps in the right direction, and they should be commended for their pioneering effort as well as for the material covered.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

PERMISSIBLE COMPOSITION AND CONCENTRATION OF IRRIGATION WATER

Discussion

BY E. B. DEBLER, M. AM. SOC. C. E.

E. B. DEBLER,[12] M. AM. SOC. C. E. (by letter).[12a]—A timely contribution on a subject of increasing importance in the field of irrigation has been presented by Professor Kelley. The relationships of the various factors, which determine the permissible composition and concentration of water that may be used safely for irrigation, have been illustrated clearly, stressing the fact that saline water is not alone in contributing to salt injury of soils. Drainage has long been recognized as playing a major rôle in determining the extent to which salt accumulation takes place. The other factors discussed, such as type of crop, climate, and method of applying water, simply serve to fix the quantity of water that must be applied to maintain a concentration in the soil solution within a range favorable for plant growth.

Rightfully, the author emphasizes the fact that in order to maintain a favorable concentration of the soil solution, the movement of dissolved salts must be maintained in a downward direction. This is not possible unless adequate drainage exists and sufficient water is applied in excess of crop use and evaporation loss to provide percolation through the soil profile. Cases are on record in which water containing 4,000 ppm and more of dissolved solids has been used continuously without any apparent deleterious effect on plant growth. However, this has been accomplished through copious irrigations on permeable soils where well-developed drainage systems exist.

A common error among irrigators has been the use of insufficient water where the salinity content was high. This practice prevented leaching and consequently resulted in excessive accumulation. Use of sufficient water to insure thorough leaching of the root zone is not interpreted as being a wasteful practice, but rather a necessity for permanence in successful irrigated agriculture.

Professor Kelley is to be commended on his presentation. It brings to the field of civil engineering information that is basic in the success of irrigation enterprises.

NOTE.—This paper by W. P. Kelley, Esq., was published in April, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1940, by Messrs. Carl S. Scofield, Walter W. Weir, and Robert S. Stockton; and November, 1940, by Leon D. Batchelor, Esq.

[12] Hydr. Engr., U. S. Bureau of Reclamation, Denver, Colo.

[12a] Received by the Secretary January 2, 1941.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MASONRY DAMS
A SYMPOSIUM

Discussion

BY JAMES S. LEWIS, JR., ASSOC. M. AM. SOC. C. E.

JAMES S. LEWIS, JR.,[120] ASSOC. M. AM. SOC. C. E. (by letter).[120a]—The information contained in the comprehensive paper by Messrs. Paul and Jacobs might well be included in the curriculum of every engineering school. Too frequently, engineers confronted with foundation problems must secure such knowledge by painstakingly arriving at solutions that have been obtained previously by other engineers. Much time is thus lost and expensive effort needlessly duplicated. Messrs. Paul and Jacobs have made a valuable contribution to a technical subject that does not lend itself to expression in exact terms. Despite certain widely applicable fundamentals, every foundation presents questions which may not be answered until the true nature of the rock is revealed by excavation.

Where the improvement of bearing strength is a serious consideration, it is frequently advisable to extend the excavation, under the gravity section of the structure, below any large horizontally disposed seams that would require filling in order to support the superimposed load. In addition to a tendency to become smaller, seams at greater depths have a tendency to pinch out and form contact areas with greater frequency, with the result that the inherent bearing strength of the foundation ordinarily becomes greater and depends less upon grouting as the depth increases.

Although natural irregularities frequently afford a condition which makes additional bond unnecessary, this is not always true, and, as Messrs. Paul and Jacobs have indicated, shale may pose a particularly difficult problem in this respect. The navigation lock of the Watts Bar Project rests upon a soft, blue, fissile shale, which contains thin layers of interbedded sandstone. In most

NOTE.—This Symposium was published in May, 1940, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: September, 1940, by Messrs. William P. Creager, J. R. Shank, George R. Rich, Robert A. Sutherland, Ross M. Riegel, Paul Baumann, W. A. Perkins, L. J. Mensch, and Lewis H. Tuthill; October, 1940, by Messrs. F. A. Nickell, Leslie W. Stocker, Barton M. Jones, P. E. Gisiger, Joseph A. Kitts, S. O. Harper, and R. F. Blanks; November, 1940, by Messrs. Berlen C. Moneymaker, A. Warren Simonds, and W. J. E. Binnie; December, 1940, by Homer M. Hadley, Assoc. M. Am. Soc. C. E.; and January, 1941, by Messrs. I. Nelidov, and James B. Hays.

[120] Constr. Supt., Watts Bar Dam and Steam Plant, TVA, Watts Bar Dam, Tenn.

[120a] Received by the Secretary January 1, 1941.

areas under the lock, the dip was found to be quite gentle, and in order to obtain bond it was necessary to trim the surface of the shale to form a series of serrations running parallel to the strike of the rock. The profile of a line taken normal to the strike would have an irregular saw-toothed appearance, with the height of the teeth varying from approximately 6 in. to 3 ft. This work was done by hand, and the use of power tools was held to a minimum to avoid disturbance of the remaining rock.

Drains formed by half-round pipe or inverted wooden troughs on the surface of the foundation rock may be plugged with mortar when covered with concrete, unless exceptional care is exercised, as vibration of the concrete reduces the mortar to a highly plastic state, enabling it to flow through extremely small openings. Relatively shallow drain holes, drilled through the concrete later to supplement the built-in drainage system, offer good insurance against excessive uplift and may be found especially valuable under aprons or other thin sections.

Messrs. Paul and Jacobs have emphasized the importance of skilled interpretation of the information obtained from core borings, and the writer firmly believes that the best talent obtainable for this purpose is the cheapest. All too frequently, poorly paid, unqualified men, lacking in interest, training, and experience, are used as inspectors of core drilling operations. The result is that they serve as nothing more than recorders, and the records are of little value. One trained engineering geologist is of greater value than a dozen such men, and the wages of a qualified man will be repaid many times over later when construction is under way.

For exploratory work, the diamond core drill is far superior to the shot drill. The core loss is less, and that core which is recovered retains its original appearance to a greater degree. As the diamond drill is superior to the shot drill, so is the double-tube diamond core barrel superior to the single-tube barrel.

Blasting for rough excavation should be carefully controlled in order to reduce the expense attendant upon the removal of damaged rock outside the required line of excavation; Messrs. Paul and Jacobs recommend that drilling be limited to a specified proportion of the total depth in order to avoid damage to the rock below. The reference is probably to operations which involve the drilling of large areas. It is frequently practical to develop a vertical face, or series of faces, in the area of excavation, so that the force of the blasts is relieved outwardly with no damage to the rock below. When this can be done, the blast holes may be drilled the full depth of the proposed excavation with no adverse results.

The shale which composes the foundation of the Watts Bar Project navigation lock was excavated with power shovels, without resort to blasting, to within limits set 1 ft above contemplated grade and 1 ft inside of final lines. The unit cost of this rough excavation was little more than that of digging compact earth, but, by the time the hand work that was required to remove the remaining surplus material was completed, the unit cost had increased several fold to a figure which was representative of the cost of removing hard rock.

It is timely that Messrs. Paul and Jacobs should point out that line drilling

is an effective method of accurately confining excavation to prescribed limits, when this is necessary, and that it is also expensive. The writer would like to emphasize the last portion of this statement and to call attention to the fact that the cost of line drilling easily may exceed the presumed value of any concrete saved by preventing overbreakage of the rock. When design considerations dictate the need for a clean, vertical face, line drilling is unquestionably justified, but considerable discretion should be exercised in specifying it for the purpose of effecting economies. Much line drilling has been done to no effective purpose.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MAXIMUM PROBABLE FLOODS ON PENNSYLVANIA STREAMS

Discussion

By Messrs. Gordon R. Williams, and Emil P. Schuleen

Gordon R. Williams,[15] Assoc. M. Am. Soc. C. E. (by letter).[15a]—For its original treatment of the problem under consideration, and for the large quantity of supporting data, this paper is a notable contribution. In this discussion the writer would like to offer suggestions for refining the procedure and to present supplemental data where possible.

The greatest weakness in the general procedure appears to be in the use of the published records of 24-hr rainfall without correction for actual duration of rainfall. Studies made in recent years, by the U. S. Engineer Department in cooperation with the Hydrometeorological Research Section of the U. S. Weather Bureau, reveal the importance of determining actual durations of rainfall in estimates of maximum probable floods. A description of the method of conducting these studies is described in a paper[16] by G. A. Hathaway, M. Am. Soc. C. E. Briefly, the procedure is to determine from observers' original records the time of beginning and ending of rainfall, and from this information to construct mass curves of total rainfall. The mass curves for the non-recording gages are adjusted where necessary by comparison with mass curves from available recording gages and estimates of air-mass conditions in the region. The adjusted data furnish a basis for determining, closely, actual time-area-depth relations, which in many cases are much different from the relations indicated by published records of 24-hr rainfall. It is obvious that such investigations can be undertaken on an extensive scale only by organizations having large resources and trained personnel.

In view of the fact that the actual durations of most, if not all, of the rainfall in the storms listed as lasting 2, 3, 4, and 5 days were for much shorter periods, the use of the published durations to estimate maximum probable rainfall for short durations will lead to very large errors. As the results of the aforemen-

Note.—This paper by Charles F. Ruff, M. Am. Soc. C. E., was published in September, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1940, by Messrs. Joseph L. Benson, H. Alden Foster, and Edgar E. Foster, and January, 1941, by C. S. Jarvis, M. Am. Soc. C. E.

[15] Associate Hydr. Engr., U. S. Engr. Office, Baltimore, Md.

[15a] Received by the Secretary January 6, 1941.

[16] *Transactions*, Am. Geophysical Union, Part II, July, 1939, pp. 195–203.

tioned studies have not been completed, information regarding actual durations of storms is very meager and is confined to point rainfalls only. Storm-duration studies of a long record at Boston, Mass., have been published by Charles W. Sherman,[17] M. Am. Soc. C. E.

As a generalization it may be stated that most of the rainfall from great storms in Northeastern United States has fallen within a period of about 60 hr or $2\frac{1}{2}$ days. Published records may indicate that the period of intense rainfall lasted as long as 5 days. It is apparent that if the total rainfall is applied to unit graphs in 5 days instead of $2\frac{1}{2}$ days there will be a great difference in the resulting flood discharges.

The U. S. Engineer Office in Baltimore has made an extensive study of seasonal rainfall at stations in the Susquehanna River Basin in Pennsylvania in connection with drainage problems for leveed areas. It may be safely assumed that the results of these studies of station rainfalls are comparable with the curves for 1 sq mile shown in Figs. 7 and 8. Fig. 21(*a*) shows the 100-yr depth-duration curve for the winter season (November to April, inclusive) for two recording stations in Pennsylvania compared with that of the author for 1 sq mile for standard winter storms, station 12, Ohio Axis. The curves for Scranton, Pa., and Harrisburg, Pa., are based on a frequency analysis for durations of from 5 min to 12 hr. Apparently, the curves presented by the author are too flat, and a suggested modification is shown as a broken line. Comparison of depth-duration-frequency curves with frequency curves for so-called 1-day and 2-day rainfalls indicates that the average duration of large winter rainfalls is about 16 hr for the 1-day storm and 24 hr for the 2-day storm.

Comparison of the curve for 1 sq mile for standard summer storms, station 8, Atlantic Axis, shown by the author, with an envelop depth-duration curve for Pennsylvania and northern Maryland is shown in Fig. 21(*b*). Again the curve given in the paper appears to be too flat.

The comparison of standard flood peaks with Fuller's formula is not significant except in a historical sense. The late Weston E. Fuller and his associate, the late Allen Hazen, Members, Am. Soc. C. E., were pioneers in the field of flood estimates, and they laid the foundations for much of the present development, especially in statistical analysis. However, more than 25 years have passed since the original Fuller paper was published (20),[17a] and in that time a mass of flood data, which would materially modify the original results, has been collected. The Fuller factor $\left(1 + \frac{2}{A^{0.3}}\right)$ was based on only twenty-six records and, in the experience of the writer, cannot be substantiated today. The complete formula is intended to give the most probable flood to be experienced in time T and not the flood to be equaled or exceeded, as do most frequency curves. Complete explanations of the Fuller method are available in a report of the U. S. Geological Survey.[18]

[17] *Civil Engineering*, March, 1939, p. 179.

[17a] Numerals in parentheses, thus: (20), refer to corresponding numbers in the Appendix of the paper.

[18] "Floods in the United States—Magnitude and Frequency," *Water Supply Paper No. 771*, U. S. Geological Survey, 1936, pp. 41–43, 46, 398–403, and 414.

As more data become available on infiltration rates, it will no doubt become advisable to substitute the method of subtracting infiltration from rainfall instead of using a runoff coefficient in determining the difference between rainfall and runoff in the maximum probable flood. Such a method is analogous

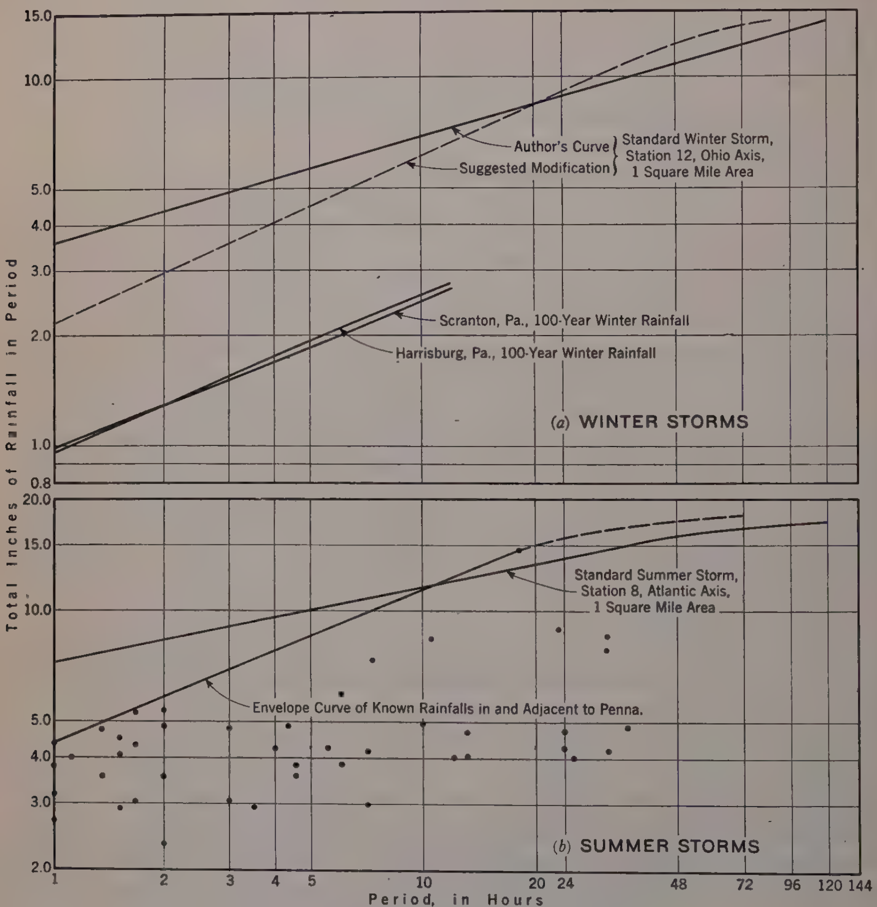


FIG. 21.—COMPARISON OF RAINFALL DURATION CURVES

to assuming a variable runoff coefficient throughout a flood and is more representative of actual flood conditions. Use of the infiltration method will tend to give higher peak discharges than the coefficient method.

It is the opinion of the writer that the relations between peak discharge and drainage area for the standard floods as shown in Fig. 14 give too flat a curve. That is, for areas less than about 200 sq miles the peak discharges are too small and for more than about 5,000 sq miles they are much too large. A curve represented by the equation $Q = \dfrac{5{,}000}{\sqrt{A}}$ closely represents an envelop curve of

maximum known rates of discharge per square mile in Northeastern United States for areas of more than about 10 sq miles. Such a curve is defined by thousands of record-years[19] and, although higher discharges probably will be recorded many times, the slope of the line should be fairly well defined. A possible explanation of the flatness of the curves in relation to actual records, especially pronounced in Fig. 19, may be that the synthetic unit graphs are not applicable to the wide range of areas considered. The matter of the actual durations of rainfall, already discussed, will affect the composition of the standard storm and in turn affect the standard flood and perhaps the relation between the standard flood peak and drainage area. It might be expected that the curve for the maximum probable flood would be closer to the actual records for the large areas because the percentage differences between rare floods usually vary inversely as the sizes of areas.

The author expresses the belief that the maximum probable flood relation which he has derived represents floods that have an equal probability of occurrence throughout the range of areas considered, and therefore the derived curves shown in Figs. 18 and 19 would be flatter than an envelop curve for maximum known discharges. There is little but abstract reasoning to confirm or dispute such a conclusion. In this connection the writer has had occasion to plot curves representing the relation between discharges of equal frequency and size of drainage area for similar regions in Pennsylvania. These curves indicate that discharges of equal frequency vary between about the 0.5 and the 0.8 power of the area, with a tendency for the more infrequent floods to vary as the smaller power of the area. The sum total of hydrologic experience is still too limited to draw any reliable conclusions as to the possible relation between curves representing discharges of equal frequency and envelop curves of maximum recorded discharges.

The writer feels that there is opportunity for a vast amount of research in this very interesting problem of estimating maximum probable floods and that the author of this paper has laid the foundations for some valuable future developments.

Emil P. Schuleen,[20] Assoc. M. Am. Soc. C. E. (by letter).[20a]—Ingenious methods are described in this paper for indicating the manner in which various factors controlling rainfall and flood discharge affect the magnitudes of the "maximum probable storm" and the "maximum probable flood." However, in view of the many assumptions that must be made in following the author's procedure to conclusion, his work should be more valuable in suggesting individual methods for assisting the engineer in making adjustments to such storms and floods as he may have under consideration for his particular problem than in providing direct values for the "maximum probable storm" and the "maximum probable flood."

The author's definition of the "maximum probable flood" is rather indefi-

---

[19] "Maximum Discharges at Stream-Measurement Stations," by Gordon R. Williams and Lawrence C. Crawford, Assoc. Members, Am. Soc. C. E., *Water Supply Paper No. 847*, U. S. Geological Survey, 1941.

[20] Senior Engr., U. S. Engr. Office, Pittsburgh, Pa.

[20a] Received by the Secretary January 10, 1941.

nite and raises the question: Just what is the "flood so large that the chance of its being exceeded is no greater than the hazards normal to all of man's activities"? His definition also suggests that he considers the term "maximum probable flood" as synonymous with the term "design flood." The writer believes that these two values should be considered separate and distinct, although the latter may be equal to the former in the case of large and important structures. The "maximum probable flood" should be the largest flood that is considered to be reasonably possible in the particular watershed, giving due weight to the basin characteristics and all other factors that would affect the magnitude of such a flood.

The "maximum probable flood," conceived in this manner, might then be used by the engineer as a basis for exercising his judgment regarding the design of the structure. For example, in the design of a channel improvement, the waterway opening under a bridge, or the spillway of a small and unimportant dam, the cost of providing discharge capacity sufficient to accommodate the "maximum probable flood" might well be prohibitive. On the other hand, in the design of a spillway for a large and important dam, where failure would result in loss of life and heavy property damage, the capacity should be sufficient to discharge, without danger to the structure, the greatest flood that might be considered reasonably possible in the particular watershed. The flood to be used as a basis of design, therefore, should be one resulting from a balance between the cost of providing a certain degree of discharge capacity and the danger in loss of life and property damage should failure of the structure occur. Such a criterion would naturally lead to the use of a flood equal to or closely approaching the "maximum probable" as a basis of design for a large and important project, but would permit the use of a flood of smaller magnitude in designing a less important project, where the hazard due to possible failure of the structure would be of minor consequence.

The author's study of variation in rainfall with respect to location along the two axes as reflected in Figs. 1 to 5, inclusive, should serve as a warning to engineers who might otherwise desire to translate storms over great distances without adjustment—as for example, from the Gulf or south Atlantic coasts to Pennsylvania. Since floods are the result of rainfall, the same warning applies equally well to the assumption that floods that have occurred in one region might occur in the same proportions in a basin of the same size and shape in another region widely removed from the first by distance or some physical feature, such as a high mountain range. As an additional caution in this respect, it might be mentioned that comparative studies of a large number of floods which have occurred somewhere in a region of relatively large extent will disclose several of unusual proportions for watersheds of their respective areas and shapes. Frequently, such records represent rugged basins with narrow valleys, steep hillsides, and steep stream gradients, which are particularly conducive to floods with high peak discharges. Records from such basins should not be considered representative of a flat or lightly rolling basin with wide valleys and sluggish streams, even though the areas and general shapes of the watersheds may be the same.

The absence of intense summer storms along the Ohio Axis in the vicinity of station 8 may be the result of the deflection of storms originating in the east Gulf or along the south Atlantic Coast by the higher mountain peaks along the Tennessee-North Carolina border. However, it is noted in Fig. 5 that there is an apparent lack of data in the vicinity of station 8 covering summer storms of any magnitude. If this deficiency of data is characteristic also of the diagrams for the other values of area and duration, it may be that records from a larger number of stations in that vicinity representing a longer composite period would disclose values sufficiently great to make the enveloping lines for the Ohio Axis summer storms similar to the enveloping lines for the winter storms or the Atlantic Axis summer storms.

The rainfall values indicated by the duration curves of Figs. 7 and 8 appear to be low in comparison with records of storms that have occurred in the general vicinity of Pennsylvania. If the rainfall records used in deriving these curves represent once-a-day observations, it is quite possible that some of the storms may have been assumed to be of 24-hr or 48-hr duration, whereas the actual duration of rainfall might have been less than 24 hr with part of the precipitation being recorded as of one day and part as of the next. If this condition exists in the data, some of the storm values should be moved to the left in the diagrams and the duration curves in general thus would be raised. The possibility of such an error in assumption could be disclosed by analyses of the records from a number of stations for each storm by means of mass curves. This method of analysis, of course, assumes that the records from one or more recording rain gages in the vicinity are available to provide a "pattern," or that the time of beginning and ending of rainfall is available for one or more stations, or that the time of the once-a-day observation is not the same at each station for which records are used in the analysis.

There is no apparent reason why the curve for the Atlantic summer flood in Fig. 14 should differ considerably from the corresponding curves for winter and Ohio summer floods by continuing to vary as the 0.3 power in the range from 1,000 to 10,000 sq miles. In this respect, it is significant that the standard flood at this station as shown in Fig. 19 is considerably above the records of actual floods from drainage areas within that range.

It is noted that the curves of Figs. 18 and 19 tend to ignore actual floods which have occurred on small drainage areas. The author states that no correction was made for the runoff coefficient in plotting these actual floods. According to Fig. 17(*c*), the coefficients for the storms causing the summer floods probably were not greater than 80%, in which case the correct values for those floods in Figs. 18 and 19 would be greater than shown and therefore even farther above the standard flood lines. It is believed that these data indicate the standard flood lines to be too low for drainage areas up to about 1,000 sq miles.

This tendency to ignore or discount unusual values is also noticeable in Figs. 7 and 8, which in some cases apparently represent storms causing some of the intense floods on small drainage areas in Figs. 18 and 19. The author has indicated that many of these intense values are the result of very old records or of unofficial records of rainfall measured in buckets, etc. Although the writer

agrees that such values may not be as reliable as later or official records, he believes that, in the absence of information to the contrary, they should be considered as reasonably accurate, after such checks as the engineer finds possible have been made.

The writer wishes to emphasize the author's conclusion to the effect that his results should not be "used blindly for design of flood-control projects or spillways." Curves and diagrams, such as those presented by the author, serve as an excellent guide for preliminary design or the check of a final design; but, in the case of a large and important structure, a careful study should be made of the storm possibilities over the particular basin, and the hydrograph of the "maximum probable flood" should be determined therefrom by means of a unit hydrograph for the specific basin, taking into account a runoff coefficient which might be considered reasonably severe, as well as an infiltration rate and a basic flow which might be considered reasonably possible of occurring coincidentally with the storm. The flood so determined might then be used as a guide for design.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

RECOMMENDED PRACTICE AND STANDARD SPECIFICATIONS FOR CONCRETE AND REINFORCED CONCRETE

Discussion

By O. G. Julian, M. Am. Soc. C. E.

O. G. Julian,[41] M. Am. Soc. C. E. (by letter).[41a]—During the thirty-six years since 1904 the Joint Committee has done much valuable work and rendered a genuine service to the profession by codifying a uniform set of rules for the manufacturing of concrete and the design of reinforced concrete structures. The judicious use of these rules by engineers throughout the United States, on the whole, has produced concrete structures that are reasonably safe, although not always sightly or durable. These faults are due largely to lack of supervision rather than to the code.

The Joint Committee Report is generally considered in the light of a bible by those interested in the art of concrete. It is refreshing to note that the latest report correctly places emphasis on durability, and in this and other ways shows improvement over previous editions.

It is doubted if enough emphasis is place upon the prime requirement necessary to obtain sightly, and durable structures. This requirement entails sympathetic and omnipotent supervision on all parts of the job, by all parties concerned—the contractor, the engineer, and the testing laboratory. It has been found empirically that, with ordinary job supervision, the old (1918) strength curve introduced by Duff Abrams, M. Am. Soc. C. E.,

$$f_c' = \frac{14{,}000}{7^{w/c}} \qquad (26)$$

(in which w/c is the water-cement ratio by volume) almost exactly represents the average strength of 28-day test cylinders, but that with such supervision a plus and minus variation of 30% may be expected. For example, using $5\frac{1}{2}$ gal of water per sack of cement and approximately 535 lb of cement per cubic

Note.—This Report was published in June, 1940, *Proceedings*, Part 2. Discussion on this Report has appeared in *Proceedings*, as follows: September, 1940, by L. J. Mensch, M. Am. Soc. C. E.; November, 1940, by Messrs. John C. Sprague, and Walter R. Hnot; and December, 1940, by Edward C. Gould, Assoc. M. Am. Soc. C. E.

[41] Chf. Structural Engr., Jackson & Moreland, Boston, Mass.

[41a] Received by the Secretary January 8, 1941.

yard of concrete, the average strength of test cylinders was found to be 3,420 lb per sq in., but the range in strength was from 2,420 to 4,410 lb per sq in. With strict supervision at both the mixing plant and the placing point, it has been found that

$$f_c' = \frac{19{,}100}{7^{w/c}} \qquad (27)$$

applies with a plus and minus variation of less than 10%. Eq. 27 almost exactly represents the locus of the values given in Table 2. The only reason that strength is mentioned here is that it is believed to be a measure of impermeability and durability. Concrete vendors and contractors have been allowed to sell and install an inferior and non-uniform product for so long that it is difficult to make them understand that specifications are meant to be read, understood, and complied with; and that concrete need not be poured like pea soup but may be installed by proper vibrating, tamping, and spading.

The committee suggests two distinct sets of recommended mixes—one for use where the contractor is to contribute his skill as a concrete technician and the second for use where the engineer is the concrete technician, the contractor functioning merely as a builder. The strength values given for the first alternate are approximately 20% higher than those given for the second. It might be inferred from this that the contractor is likely to be a better technician than the engineer. In many cases this is a fact. Would it not be better to require that the contractor and engineer, in all cases, work together sympathetically with the idea of turning out a superior product instead of a product that only slightly exceeds the requirements of the specifications and in some cases may not meet such requirements? The need for a dual standard as represented by the two alternates is not apparent.

Since, with a low water-cement ratio, high-strength concrete can be produced with a low cement content, and since such concrete is likely to be permeable and not durable, it is believed that the minimum allowable cement content as well as the maximum allowable water-cement ratio should always be specified. Although the water-cement ratio is important, the use of a low water-cement ratio is far from a panacea. Proper materials, well-graded aggregates, sufficient cement, high quality workmanship, and thorough curing are also most important.

The eight mixes indicated in Table 2 appear to be excessive in number. Concrete that tests less than 3,000 lb per sq in. is (or at least should be) something which passed with the "great flood." The three classes indicated in Table 12 are believed to be sufficient for most practical purposes (dams and

TABLE 12.—RECOMMENDED MIXES [a]

No.	Description	Class A	Class B	Class C
1	Maximum allowable net water content (gal per sack cement)	5	5.5	7
2	Minimum allowable cement content (lb per cu yd)	655	535	380
3	Expected minimum compression strength at 28 days (lb per sq in.)	5,200	4,500	3,200

[a] Concrete classes are as follows: Class A, used where first class concrete is required; Class B, for ordinary building use; and Class C, for unimportant and not exposed work (concrete fill).

other unusual structures excepted). It should be stated clearly in the specifications that the "minimum allowable cement content" ordinarily will have to be increased in order to produce the desired workability and that the plasticity and workability of the concrete shall always be such that it can be vibrated, tamped, spaded, or otherwise worked into place entirely without honeycomb or other defects.

Vibrators should always be used. When used the minimum slumps given in Table 4 are believed to be ample except for cases where the reinforcement is very closely spaced, and then it may be advisable to increase the slump to 4 in. Vibrators should be capable of transmitting at least 5,000 vibrations per min when embedded in concrete to their full working depth.

The seven-day curing period specified is believed to be inadequate. The following is suggested:

> "All exposed surfaces of concrete shall be kept thoroughly moist by an approved method for at least fourteen days and as much longer as is practicable. This applies to vertical as well as horizontal surfaces. If curing is to be effected by the concrete surfaces remaining in contact with the forms, the forms shall be kept damp continuously. Concrete shall not be wetted and then allowed to dry."

The effectiveness of curing over protracted periods has been demonstrated by H. J. Gilkey,[42] H. D. Dewell,[43] Members, Am. Soc. C. E., and others.

Job curing in sisalkraft paper and with colorless curing liquid has been found to be only 51% and 26% effective respectively.[43] The proper curing of concrete is probably as important as any operation that enters into its installation; yet it is the most neglected.

The difference in cost between indifferent concrete and a superior product that is impermeable, durable, and tests 4,500 lb per sq in. or more at twenty-eight days has been found to be only about 5%, when the cost of the forms, and the reinforcing steel are included. The principal ingredient required to obtain good durable concrete is not some nostrum admixture, or more detailed specifications, but sympathetic and omnipotent supervision on all parts of the job—supervision that knows the best practice and will permit nothing but the best. The cost of such supervision on all parts of the work is the most profitable investment that can be made in the interest of obtaining durable structures.

Since the moment coefficients given in Appendix 3 are likely to be misapplied and used for cases other than those even approximating "equal span lengths, equal stiffness and uniform loading," as have similar coefficients in the past, it is believed they should be omitted.

The abbreviated Moment Distribution Method[5] given in Appendix 2 has a most limited application. Most concrete structures are more or less irregular rigid frames and are subjected to sidesway. Such frames can be readily analyzed by the unabbreviated Moment and Shear Distribution Method, full cognizance being taken of the effect of sidesway and an entire bent being

---

[42] "The Moist Curing of Concrete," by H. J. Gilkey, *Engineering News-Record*, October 14, 1937.

[43] "Testing the Efficiency of Concrete Curing Methods," by H. D. Dewell, *Western Construction News*, December, 1939 (summarized in *Journal*, Am. Concrete Inst., April, 1940, p. 529).

[5] For a more complete treatment of this method, see "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan (John Wiley and Sons, N. Y., 1932).

considered as a unit. For cases in which "no great irregularities of span, story-height or loading are involved" the unabbreviated method is so readily applied to an entire bent as to leave little excuse for the use of an abbreviated method such as that indicated in Appendix 2. On the other hand, the use of the abbreviated method on irregular bents will lead to errors of considerable magnitude.

As stated in the report, the recommendations regarding two-way slabs with supports on four sides and flat slabs apply only where certain general limitations are observed. Additional interesting and instructive analytical data on the bending of anisotropic plates and the bending of plates supported by rows of columns were published in 1940.[44] These data might be adopted advantageously in reinforced concrete design and thereby place it on a more general and rational basis.

The design of web reinforcement has long been a controversial question. Eq. 1 gives the intensity of diagonal tension at the neutral surface before the beam begins to crack. At points between the neutral surface and the tensile steel the intensity of this tension is increased, whereas between the neutral surface and the compression face of the beam it is decreased, on account of the longitudinal fiber stress. Accepting the elastic theory on which the report is based, the stress in diagonal bars is $n f_t$ (f_t = intensity of diagonal tension stress) until the concrete cracks; after which it appears self-evident that the diagonal tension must be resisted by the steel entirely or transferred to another part of the cross section. There does not appear to be any better reason for depending on concrete to resist diagonal tension than for depending on it to resist longitudinal fiber tension. The limiting value of f_t resulting from any cause, it is believed, should be zero across a crack and not more than $0.03 f_c'$ at such locations as those where the concrete can be depended upon not to crack. It is generally recognized that vertical stirrups are not stressed until the concrete has cracked. No matter how closely vertical stirrups are spaced they leave some tension to be resisted by the concrete, and on account of their greater obliquity with trajectories of stress, they are not as effective in resisting diagonal tension as are diagonal bars. The latter delay the formation of cracks, may minimize their enlargement, and increase the ultimate shearing strength to a greater extent than do vertical stirrups. These facts are indicated by theoretical considerations and are verified by tests.[45] Because of the foregoing reasons it is believed that paragraph 817(*e*) should be made to read approximately as follows:

> Where the shearing stress exceeds $0.03 f_c'$ the web reinforcement should be designed to carry the entire diagonal tension in bent up bars or properly anchored diagonal bars provided for that purpose.

With this restriction f_s for web reinforcement might be given the same value as for longitudinal fiber stress—("tension in flexural members").

Although practically all building columns are subjected to biaxial flexure and direct stress the report avoids indicating methods of calculating the stresses

[44] "Theory of Plates and Shells," by S. Timoshenko, *Engineering Societies Monograph*, Articles 37 and 46.

[45] *Technological Paper No. 314*, National Bureau of Standards, U. S. Dept. of Commerce, 1926.

for these conditions. The authors of textbooks also avoid this most important problem and confine their discussions to the combination of direct stress and flexure in the plane of a principal axis of inertia. The statement (see paragraph No. 860) that, "In reinforced concrete columns subjected to bending moments, the recognized methods of analysis should be followed in calculating the stresses due to combined axial load and bending," is therefore of little help. So far as the writer knows no direct general analytical method of calculating the stresses due to biaxial flexure and direct stress in reinforced concrete members has been published in English.

A. Roussopolis has given a general analysis,[46] in algebraic form, and charts that greatly facilitate the computation of stresses in homologous rectangular sections. The graphical analysis of this problem has been given by B. A. Rich, M. Am. Soc. C. E., and W. W. Bigelow.[47] It has been extended by A. B. Rich in an unpublished paper. Articles on this subject by Paul Andersen,[48] and by William Saville,[49] Assoc. Members, Am. Soc. C. E., are also worthy of note.

It is believed that the working methods given in these articles and the general formulas developed by Mr. Roussopolis might be abstracted and made part of subsequent reports.

The values of the modular ratio given in Table 7 apparently are meant to represent intermediate values between those pertaining to short-time loadings, such as live load, impact, or those due to temperature changes, and loadings for protracted periods, such as dead loads. The startling difference between the secant moduli applicable to short-time loading and the equivalent moduli applicable to long-time loading (which latter take into account the effect of plastic flow, shrinkage and change of elastic properties with time) has been pointed out in textbooks [50] and numerous articles in the technical press [51] published during the past decade.

It should be emphasized that, in computing stress from thrusts and moments the actual modulus pertaining to the age of the concrete, the magnitude of the stress, and the duration of the loading under consideration, should be employed. The modulus applicable to loadings of long duration, such as dead loads, may be of the order of 750,000 lb per sq in. (n = approximately 40) whereas the modulus that is more nearly applicable for live loads, impact, and temperature stresses may be about 4,900,000 lb per sq in. (n = approximately 6). These values apply for concrete having f_c' equal to approximately 3,000 lb per sq in.

Various investigators have reported widely different values for E_c and in many cases have failed to give all the pertinent data and conditions. The

---

[46] "Die allgemeine Lösung des Problems des exzentrisch beanspruchten Eisenbetonquerschnitts," by A. Roussopolis, *Annales Techniques*, Athens, May, 1933 (abstracted in *Beton und Eisen*, March 5, 1939).

[47] *Journal*, Boston Soc. of Civ. Engrs., February, 1926.

[48] *Civil Engineering*, August, 1938, p. 549; January, and October, 1939, pp. 37 and 618, respectively; and January, 1940, p. 37.

[49] *Loc. cit.*, March, 1940, p. 170.

[50] "Principles of Reinforced Concrete Construction," by F. E. Turneaure and E. R. Maurer, 4th Ed. (1932), p. 349.

[51] Discussion on "Unsymmetrical Concrete Arches" and the references given therein by the writer, *Journal*, Boston Soc. of Civ. Engrs., May, 1933. Also, "Plastic Flow and Volume Changes of Concrete," by R. E. Davis, M. Am. Soc. C. E., and H. E. Davis and E. H. Brown, Assoc. Members, Am. Soc. C. E., *Proceedings*, A. S. T. M., Vol. 37, 1937, p. 317; and "Studies in Reinforced Concrete," *Technical Paper No. 21*, October, 1939, Building Research Dept. of Scientific and Industrial Research, London, England.

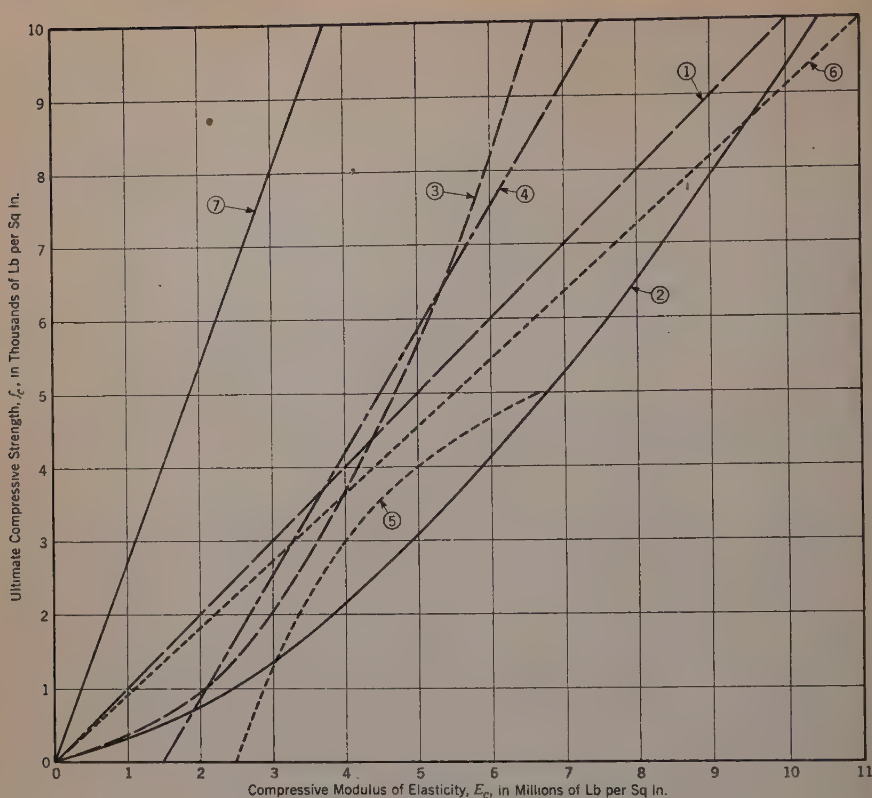


FIG. 1.—VARIATION OF E_c WITH f_c'

results of a few of these reports are summarized graphically in Fig. 1, in which the formulas for the various curves are identified as follows:

Curve No. (Fig. 1)	Formula	Reference
1	$1{,}000\, f_c'$	Joint Committee Progress Report, January, 1937, Table 8, p. 44.
2	$E_c(\text{initial}) = 33{,}000\,(f_c')^{5/8}$	"Modulus of Elasticity of Concrete," by Stanton Walker, *Bulletin No. 5*, Lewis Inst., Chicago, 1920.
3	$E_c(\text{tangent at } \frac{1}{4} f_c'; \text{ or, secant at } \frac{1}{2} f_c') = 66{,}000\,(f_c')^{\frac{1}{2}}$	(same as 2)
4	$E_c = (1.5 \times 10^6) + 600\, f_c'$	Discussion by Gilbert C. Staehle of Progress Report of Committee 312: "Plain and Reinforced Concrete Arches," *Journal*, Am. Concrete Inst., October, 1932, p. 96.
5	$E_c(\text{upon loading}) = \dfrac{20 \times 10^9}{8{,}000 - f_c'}$	Derived from "Creep of Concrete Under Load," by W. H. Glanville, *The Structural Engineer*, February, 1933, Fig. 19, p. 67.
6	$E_c(\text{one day after loading}) = 1{,}100\, f_c'$	(same as 5)
7	$E_c(\text{limiting value for long-time loading}) = 370\, f_c'$	(same as 5)

In Curve 7, E_e, which is the equivalent value of E_c required to take account of plastic flow, does not take account of shrinkage. It is believed that much work remains to be done in determining the "elastic properties," equivalent moduli of elasticity, and Poisson's ratio or concrete for various conditions. Data on the latter property appear particularly rare.

Might it not be advisable for the committee to publish the values of two sets of n, one applicable to short-time loading and the other applicable to long-time loading?

It is rather disappointing to find that the report does not cover such important subjects as: (1) The effect of shrinkage and plastic flow; (2) the treatment of composite beams; (3) methods for computing deflections; and (4) temperature gradient through concrete sections and the resultant stresses. All of these subjects are important and have been treated in textbooks and the technical press.

The entire report is based on the premise that concrete may be treated as an elastic material and this discussion so far has "gone along with" that assumption. However, it has long been known that concrete, even within the range of commonly accepted working stresses ($f_c = \frac{1}{3} f_c'$ approximately) is far from being a perfectly elastic material. It is still more plastic, of course, when worked to $f_c = 0.45 f_c'$ or to $f_c = 0.60 f_c'$ (when wind, earthquake, and other unusual forces are considered) as proposed in the report. Stresses calculated on the basis of Hooke's law being valid and shrinkage and plastic flow being negligible do not bear any consistent relation to the actual stresses or to the actual useful strength of the members. This has been shown to be true by tests on buildings, published [52] several years ago. The effect of shrinkage and the plastic properties of concrete appear to make the calculations of the actual stresses in a member, within the working range, altogether impracticable. However, studies reported in 1940 by Charles S. Whitney,[53] M. Am. Soc. C. E., clearly indicate that shrinkage and plastic flow have only a relatively slight effect upon the useful strength of a member because prior to failure there is a redistribution of stress that largely nullifies their effects.

The plastic theory has a number of advantages. Among these are:

(1) It recognizes concrete frankly for what it is—a plastic rather than an elastic material;

(2) It makes the useful strength of members closely predictable and thereby allows designs to be based on a consistent factor of permissible overload; and

(3) It is simple.

The principal disadvantage of Mr. Whitney's method appears to be that it gives no information as to the actual stresses prevailing when a member is subjected to working loads and hence no data from which to compute deflections. However, the values of such stresses computed by the conventional elastic method are likely to be misleading rather than helpful. This is espe-

[52] "Deformation of Steel Reinforcement During and After Construction," by Sergius I. Sergev, *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1343.

[53] "Plain and Reinforced Concrete Arches," by Charles S. Whitney, Author Chairman, Report of Committee 312, Am. Concrete Inst., *Journal*, Am. Concrete Inst., September, 1940, p. 1; also "Plastic Theory of Reinforced Concrete Design," by Charles S. Whitney, *Proceedings*, Am. Soc. C. E., December, 1940, p. 1749.

cially true if E_c and Poisson's ratio are not known accurately and if allowances are not made for the effect of shrinkage and plastic flow.

The report has given partial recognition to the plastic method in the adoption of the column formula, Eq. 9. It is believed that it would be a credit to the profession if it were to admit frankly its past sin—the worship of the false god "elasticity" as applied to concrete—and start anew to rewrite the code of recommended practice, in so far as the computation of stresses from a given set of moments and thrusts is concerned. It has been shown elsewhere [51, 54] that the invalidity of Hooke's law does not lead to large errors in the computation of moments, thrusts, and shears except when the effect of temperature is concerned in which case the errors may be greatly on the side of safety.

The value of the report would be enhanced if references were given in which the test data supporting, and the rationalization of the "rules of thumb," tables and formulas stated might be found. Many of these formulas are far from self-evident. Designers should not use formulas without knowing and understanding the hypotheses upon which the argument is based, understanding thoroughly all the steps in the argument, and being able to interpret the conclusion consciously. The application of a formula outside of the limits for which it was derived has often led to weird results. It may be impractical and out of place to include test data, hypotheses, and arguments in detail in the report, but there appears to be no good reason why a complete list of references should not be given as to where data, supporting the conclusions stated in the report, may be found.

---

[54] "Plastic Flow in Concrete Arches," by Lorenz G. Straub M. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 95 (1931), p. 613.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

RELIABILITY OF STATION-YEAR RAINFALL-FREQUENCY DETERMINATIONS

Discussion

BY MESSRS. C. S. JARVIS, AND HOWARD W. BROD

C. S. JARVIS,[24] M. AM. SOC. C. E. (by letter).[24a]—The station-year approach to rainfall frequencies, magnitudes, and normal patterns of distribution, intensity, and recurrence has long been accepted with some reservations and misgivings. Its use persisted because of the obvious advantages which it offered in cementing short segments and forming therefrom a synthetic record of impressive length and plausibility, with evident claims to reliability that could not be readily disproved.

The service performed by the author in making the exhaustive analysis, rigorous tests, and logical findings cannot be appraised adequately until one explores the labyrinths of disjointed, intermittent, and fragmentary records that have heretofore been set aside, much as is the custom for segregating refractory ores, in the hope that processes may yet be devised for their reduction and the sure recovery of their intrinsic values. Such a process is now available, largely as a result of this paper.

One must recognize, however, that more than one method is used for deriving frequency of rainfall. This subject is discussed under the heading "Introduction," where five occurrences in excess of a given amount within a record comprising 500 station years are regarded as "equivalent to an average frequency of 100 years per occurrence, which is the same result that would be obtained by taking the fifth highest amount in a station-year record of 500 years." Here is where another method has been used, along the following lines of reasoning: The maximum value occurred once in 500 years; the second from the top value, once in 250 years; the third is likely to occur once in 125 years; the fourth, once in 62.5 years; and the fifth, once in 31.25 years. The question is, which of the two methods should prevail, or is there a third approach?

NOTE.—This paper by Katharine Clarke-Hafstad was published in November, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1941, by Paul V. Hodges, M. Am. Soc. C. E.

[24] Hydr. Engr., SCS, Dept. of Agriculture, Washington, D. C.

[24a] Received by the Secretary December 20, 1940.

Should one recognize the zone between the values of 100 and 31.25 as being the range of uncertainty regarding probable frequency?

As it happens, nearly a comparable variation in results occurs under the heading, "Use of N_d for Actual Rainfall Data in Determining Standard Errors of Frequencies of Rainfall Amounts," in answer to the writer's question of some years ago. There, the

> "* * * 300 station years (15 stations for 20 years each) * * * would be more nearly equivalent to 5 × 20 or 100 years; one might call this the valid station years of record based on the average number of stations affected per storm. For the foregoing reasons it would be still better to use the value N_d rather than N_a. This would give 36 valid station years for rainfalls of one inch for the Carolina quadrangle."

Does this mean that the best one can do is to bracket the most probable values between those limits?

It was the writer's good fortune to view the early results of the author's current research, some years ago, and to utilize some of the findings to good advantage. Thereafter, the writer urged the need not only for further exploration and tests, but also for their prompt release to the profession through technical publications, such as have been chosen. The paper seems to present much data, many tests, and a certain maturity of concept beyond the tentative and informal accounts of research to which the writer had access some years ago. Now, having observed the modest beginnings, the step-by-step development, and the seemingly authoritative analyses and solutions of successive problems, and having tested them by the practical standards that have been acquired throughout a varied experience in hydrologic fields, the writer indorses the paper as a distinct scientific achievement.

Perhaps a lack of perception will account for the writer's inability to reconcile some minor details thus far, but he anticipates that these matters will be cleared up in the author's closing comments.

HOWARD W. BROD,[25] JUN. AM. SOC. C. E., (by letter).[25a]—It is often more important to understand the limitations of a law than to know the law itself. The author unmistakably implies that the station-year method is applicable to large drainage areas. It is significant to point out that the Miami Conservancy District confined the use of the station-year method to the "* * * design of sewer systems, bridge and culvert openings for small drainage basins, and in the design of dams, levees, and channel improvements where the watersheds involved are not more than a few square miles in area."[26] Furthermore, the practice of grouping stations with markedly different rainfall characteristics is questionable.[27]

There are two fundamental weaknesses in the station-year method which restrict its applicability to a very limited class of data: (a) The variation of seasonal rainfall; and (b) the dissimilarity of rainfall stations.

---

[25] Assoc. Hydr. Engr., Chg. Hydrology Studies, 3d Locks Project, The Panama Canal, Hydraulics Section, Special Eng. Div., Diablo Heights, Canal Zone.

[25a] Received by the Secretary January 6, 1941.

[26] *Technical Buletin No. 5*, Miami Conservancy District.

[27] "Rainfall Intensities and Frequencies," by A. J. Schafmayer, M. Am. Soc. C. E., and B. E. Grant, *Transactions*, Am. Soc. C. E., Vol. 103 (1938), p. 344.

(*a*) *Variation of Seasonal Rainfall.*—First, consideration will be given to the variations in seasonal rainfall quantities. From the interpretations that have been made of the longest available records of floods (Nile River, Egypt), it is evident that no secular or long-time changes in climatic conditions are taking place. Cyclic variations, however, are an integral characteristic of seasonal rainfall. There seems to be an inertia in the unbalance of the atmospheric forces which causes prolonged series of wet seasons and dry seasons to take place. The results that would be obtained if the station-year method were applied to a set of data which lies wholly within a wet phase or wholly within a dry phase might lead to tragic consequences, such as a dam failure or a shortage in water supply. Reference is made to the case cited in a National Resources Committee report[28] of a federal project which was based on a 20-yr hydrologic record. "Unhappily, the years of record seemed to have coincided with a wet period in the precipitation cycle. When the dry period came the project could be developed to only one-third its intended size."

The relevancy of the cyclic variations of seasonal rainfall to the occurrences of high rates of rainfall is readily apparent to those who have tried to correlate major floods with annual rainfall. Although it is possible to have a major flood during a comparatively dry year, it would be the exception rather than the rule; by far the greater number of major floods occur in comparatively wet years. Similarly, a greater number of high rates of rainfall will take place during wet years than during dry years and, therefore, such occurrences will parallel the cyclic variations of seasonal rainfall.

(*b*) *Dissimilarity of Rainfall Stations.*—On a purely theoretical basis, the station-year method is invalid because it assumes that there is equal opportunity for every station to receive an equal quantity of rain. Of course, this is not true since the characteristics of rainfall stations are similar to those of fingerprints in that no two of them are alike. No two rainfall stations can have the same conditions of geographic location, elevation, slope, exposure, distance from the ocean or other large body of water, etc., and, therefore, the rain-catching potentialities of any two rain gages cannot be exactly alike.

The station-year method is valid only when the data do not conflict with the limitations made in items (*a*) and (*b*). Condition (*a*) may be satisfied by developing an "index" of mean seasonal rainfall for the region for a sufficiently long period of time to assure a reliable approximation of the mean seasonal rainfall. This may be accomplished satisfactorily on the basis of a single long-term record, but preferably more. If no long-term records are available in the region, a meteorologic study of the storm paths that are characteristic of the region should be made, and the characteristics of the "index" of mean seasonal rainfall of the adjacent regions which influence the rainfall of the region under consideration should be transferred to that region. Condition (*b*) may be considered negligible in small areas of comparatively uniform meteorologic characteristics and in certain regions where the differences in the areal distribution of rainfall quantities are equal to or less than the error with which a rain gage represents the average rainfall over the area in the vicinity of the rain gage.

[28] "Deficiencies in Basic Hydrologic Data," National Resources Committee, September, 1936.

There are several other phases of the subject of rainfall probabilities upon which the author touches which provoke discussion. Among these the determination of dependence or independence of the events or observations is of prime importance. Eugene L. Grant, M. Am. Soc. C. E., has pointed out that one of the weaknesses of the flood-probability method lies in the fact that "the successive floods on a given stream in a given year are certainly not independent events." The same is true regarding successive rainfall events of the frontal type. When a series of fronts (for example, cold fronts) invade a region in which warm air has been stagnating, the first cold front meets a great opposition in meteorologic characteristics. With the passage of subsequent fronts, the meteorologic characteristics of the region become modified and the differences between the invading air and the residual air become less. The phenomena are similar to the change in saturation conditions for successive floods. The meteorologic conditions often associated with major floods consist of a series of fronts passing over a region. The successive storms associated with the passage of these fronts are certainly not independent events.

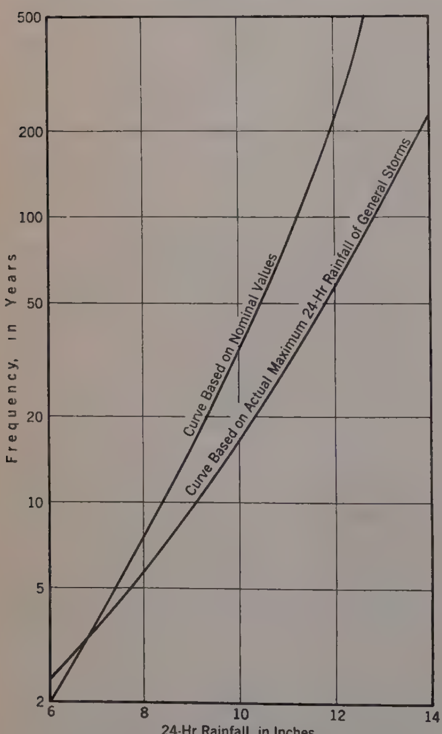


Fig. 3.—24-Hr Rainfall Probability Curves for Cristobal, Canal Zone

The principal reasons for the superiority of rainfall probabilities over flood probabilities are that the aforementioned dependency is true for only a small part of a rainfall record and that rainfall is a function of much fewer variables and therefore requires less data to establish the position of the frequency curve.

The inadvisability of using the nominal, or published, 24-hr rainfalls as the basis of a frequency study is indisputable. This practice violates the requirement of independency of events and also brings up the question of "How great, in the long run, is the difference between nominal and actual maximum 24-hr rainfall?" The independence requirement of the rainfall events is violated when the end of the nominal 24-hr rainfall comes within the storm period and a single event is indicated as two occurrences. The differences in the results that may be obtained using nominal or published 24-hr rains are shown in Fig. 3. The example has been selected at random, and therefore it is probable that even more serious differences exist.

The subject of rainfall probabilities is of such great importance to the engineering profession that it would undoubtedly prove profitable for some organ-

ization to conduct an investigation of rainfall probabilities for various locations throughout the world. The results of an analysis and correlation of such data should prove of unlimited value. As the author points out, the introduction of the area factor into frequency studies of rainfall has not yet been made and is an outstanding deficiency in hydrologic research. The work of the hydrology organizations of the U. S. Soil Conservation Service and Tennessee Valley Authority is directed toward overcoming this deficiency and should represent a great advance in the application of probability technique to rainfall data.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

PLASTIC THEORY OF REINFORCED CONCRETE DESIGN

Discussion

By Messrs. L. E. Grinter, and Basil Sourochnikoff

L. E. Grinter,[38] M. Am. Soc. C. E. (by letter).[38a]—The procedure for the structural design of reinforced concrete that is proposed in this paper is certain to receive repeated and probably favorable consideration in the coming years. There is reason to question if the knowledge of the action of reinforced concrete frames under ultimate loads is adequate today to answer the question as to whether the author's procedure is logical in all cases; and, of course, the proposed method of allowing for plasticity is but one of many possible procedures. It is true, however, that designers have already taken certain steps in this direction—notably, the adoption of a column formula that omits consideration of relative moduli of elasticity by basing load capacity upon the yield strength of the steel. On the other hand, this step was a much more obviously acceptable one in that high steel stresses in columns are known to exist as a normal condition even under ordinary working loads. Elimination of the term n in present formulas for the design of reinforced concrete beams presupposes flow beyond conditions ordinarily normal in concrete structures.

The author's castigation of the use of the ratio n in reinforced concrete design nevertheless seems justified by many recent findings. It is true that measured strains in concrete cannot be transferred into stresses of great significance, and yet one cannot question that the stress corresponding to a given strain will bear a relation, however complex, to the modulus of elasticity. The directness of this relationship may be expected to reduce as the load is increased beyond working conditions. As shown clearly by Fig. 1, the straight-line relationship between stress and strain may be expected to exist up to one half of the ultimate strength, and engineers do not plan to design structures for working stresses of even this magnitude. On the other hand, Fig. 1 does not express the stress-strain relationship as it is influenced by time yielding since this influence seems largely restricted to dead-load conditions.

Note.—This paper by Charles S. Whitney, M. Am. Soc. C. E., was published in December, 1940, *Proceedings*.

[38] Vice-Pres. and Dean, Graduate School, Illinois Inst. of Technology, Chicago, Ill.

[38a] Received by the Secretary January 10, 1941.

Evidently, then, the acceptability of any method of design by means of a theory of plasticity should be studied in the light of one's belief in design for failure. After most careful study, the writer has concluded that, in many instances, a steel structure can be designed far more logically if failure is accepted as the basis. By failure is meant a condition of deformation dependent upon the passing of the elastic limit, not necessarily a condition of exaggerated distortion. Similarly, in reinforced concrete design, it would seem logical to design the structure like the famous "shay" so that it would fail simultaneously in all of its parts. A criterion, then, is whether the proposed plastic theory will achieve this condition more closely than use of the ordinary straight-line formulas for the design of reinforced concrete beams.

In order that a continuous frame may fail simultaneously at many places (and nearly all concrete structures are continuous frames of some type or other), the designer must not only consider plasticity in the design of the individual sections but also plasticity for the distribution of moments. This is precluded in the author's procedure by his belief that "* * * the theory of elasticity is the best guide to the forces acting on the different sections." It is precluded in the writer's opinion by the simple fact that engineers do not know enough about the influence of plasticity upon structural action to predict properly the influence of time yield or of plastic flow upon moments.

The other point that deserves consideration is the fact that the parts of a structural frame include columns, beams, slabs, and their joints. To be better than the present methods, a new theory must predict the failure of each of these parts more accurately than standard procedures, and this must be true even if concrete of different strengths (and different moduli of elasticity) are used in the several parts of the structure. In early attempts at analysis, the factor n was introduced consistently for columns, beams, and slabs, and perhaps within reasonable limitations designers achieved uniformity of action under light loads for which elastic conditions obtained. The realization, however, that column rods picked up additional stresses through time flow in the concrete spoiled this apparent consistency and has led the author to attempt a readjustment of beam design to parallel the now accepted method of column design. Some revision of beam design seems desirable, one reason being that the factor of safety is probably greater in beam design than in column design by present methods. A problem for consideration is that present column formulas are justified by time yielding, whereas the proposed beam formulas depend upon plastic flow near failure. Evidently these are quite different conditions.

As was mentioned earlier, there are many ways that plasticity can be introduced into design. One procedure would be to accept the parabolic stress distribution curve seriously proposed in the early days of reinforced concrete. The author's rectangular stress pattern, being non-physical in its aspects, can be justified only by simplicity, and, actually, it does not seem to simplify, greatly, the common methods of design. Hence, although the writer agrees with the probable desirability of considering failure as a criterion for design, he finds it difficult to justify a procedure resulting in an entirely artificial stress-

distribution relationship for the concrete above the neutral axis. It seems unfortunate that one dimension of the stress diagram does not represent the maximum concrete stress and that the other dimension is not associated with the depth of the beam in a physically significant manner.

It must not be forgotten, when theories are proposed for introduction into standard codes, that these theories must be presented to undergraduate students. It is always feasible to teach facts that may be substantiated in any reasonable way. It is also possible to present hypotheses that have some reasonable connection with observed phenomena and to develop theories from such hypotheses. It is simply not reasonable, however, to ask students to accept theories that do not follow physically observable phenomena or even the accepted hypothesis as to physical action. An example of this is the possible use of a stress diagram for compression in the concrete that is inconsistent with the hypothesis of a straight-line variation of strain throughout the depth of the beam. Such an inconsistency would militate against the logical use of any formulas that might be developed, irrespective of their simplicity or other desirable features.

The author's paper will serve a most valuable purpose in pointing out the present inconsistencies in the design of reinforced concrete structures. His proposal will call forcefully to the attention of structural engineers the desirability of basing design procedures for all members of a structure upon a single hypothesis. The importance of plasticity both as an influence upon the stress distribution at a section and upon the moment distribution throughout the structure will be reweighed as to its importance. Time yielding at low loads will be reconsidered and compared in its significance with plastic flow beyond normal working stresses. The many possibilities of stress distribution over the beam cross section may be reconsidered again to advantage. Finally, the purpose to be achieved by any theory of stress analysis that is known to be far from exact in its representation of conditions in a material that is neither homogeneous nor even continuous must be re-evaluated. The author's excellent paper will bring each of these matters again to the attention of structural engineers.

Basil Sourochnikoff,[39] Esq. (by letter).[39a]—In order to simplify computations, Mr. Whitney replaces the curvilinear stress distribution with a rectangular stress block. This appears to be quite satisfactory, as the results of formulas obtained by using this approximation check very closely with experimental data. However, it may be of interest to establish formulas based directly on the curvilinear stress distribution without using any simplifying device. Of course, these formulas cannot be expected to be more exact than the stress-strain curve on which they are based. A comparison of results thus concluded with the experimental data may show just how closely the idealized stress-strain curve represents actual conditions.

Under-Reinforced Beams.—From the conditions of equilibrium of any cross

[39] Registered Structural Engr., St. Paul, Minn.

[39a] Received by the Secretary January 14, 1941.

section it follows that

$$\kappa = \frac{f_s'}{f_c'} \times \frac{p}{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (38a)$$

and

$$\frac{M}{f_c' \, b \, d^2} = \kappa \, A \, [1 - \kappa \, (1 - \bar{x})] = \frac{f_s}{f_c'} \, p \left[1 - \frac{f_s'}{f_c'} \, \frac{p}{A} \, (1 - \bar{x}) \right] \ldots (38b)$$

in which, in addition to the author's notation, f_s' is the yield stress in steel. When the steel is strained above the yield point, f_s' remains practically constant and independent of ϵ'. Then, for each particular beam, the resisting moment will vary with $\frac{1 - \bar{x}}{A}$ alone. The maximum resisting moment will correspond to the minimum of $\frac{1 - \bar{x}}{A}$. This last quantity, for each given grade of concrete, depends on the stress curve only. For curves of Fig. 3:

1,000 ϵ	$\frac{1 - \bar{x}}{A}$
1.0	0.710
1.5	0.585
2.0	0.540
2.5	0.538
3.0	0.538
3.5	0.543
4.0	0.575
4.5	0.602
5.0	0.638

Therefore, if the steel is overstressed, the resisting moment increases until the strain in the concrete reaches a value of about 0.00275. Then it would decrease if the strain exceeds this value. The ultimate strain in the concrete is therefore about 0.00275 and does not depend on the percentage of steel. The ultimate resisting moment is determined from the expression:

$$\frac{M}{f_c' \, b \, d^2} = \frac{f_s'}{f_c'} \, p \left[1 - 0.538 \, \frac{f_s'}{f_c'} \, p \right] \ldots\ldots\ldots\ldots (39)$$

which is of the same form as Eq. 7*b*. The coefficient $\frac{m}{2}$ in Eq. 7*b* is $\frac{f_s'}{2 \times 0.85 \, f_c'} = \frac{0.59 \, f_s'}{f_c'}$. For $f_c' = 4.000$, $f_s' = 6.000$ and $p = 3\%$. This means a difference of only 3.2% on the ultimate moment. Assume that, at the yield point, the steel is strained to 0.002. Since, in the present case, the steel is overstressed, $\epsilon' > 0.002$. Then, if the concrete fails after the steel is overstressed, failure occurs at $\frac{\epsilon'}{\epsilon} > \frac{0.002}{0.00275}$ or 0.73. From Eq. 38*a* this is possible only if $p < \frac{0.46 \, f_c'}{f_s'}$. Therefore, in the present case, it seems that $\frac{0.46 \times 4{,}000}{60{,}000}$ or 3% is the limiting ratio of reinforcement. The beam would be under-

reinforced if $p < 3\%$, and over-reinforced if $p > 3\%$. This again checks with results obtained by Mr. Whitney.

Over-Reinforced Beams.—From the conditions of equilibrium it follows again that:

$$f_c' A \frac{\epsilon}{\epsilon + \epsilon'} = E \epsilon' p \quad \text{.....................} (40)$$

Therefore,

$$\frac{\epsilon'}{\epsilon} = \frac{1}{2}\left(\sqrt{1 + \frac{4 f_c' A}{E p \epsilon}} - 1\right) \quad \text{.................} (41)$$

and

$$C = \frac{M}{f_c' b d^2} = A \kappa [1 - \kappa (1 - \bar{x})] \quad \text{..............} (42)$$

Using again the curves of Fig. 3, the constants for different steel ratios are given in Table 8. This shows that the ultimate resistance occurs for $\epsilon = 0.003$,

TABLE 8.—Constants for Various Steel Ratios

1,000 ϵ	Steel Reinforcement (for $p = \infty$—$\epsilon' = 0$; $\kappa = 1$; and $C = A\,\bar{x}$)									
	3%			4%			5%			∞
	1,000 ϵ'	κ	C	1,000 ϵ'	κ	C	1,000 ϵ'	κ	C	C
1	1.105	0.474	0.203	0.905	0.525	0.220	0.78	0.563	0.233	0.330
2	1.75	0 535	0.310	1.44	0.582	0.330	1.21	0.624	0.346	0.442
3	2.10	0.590	0.352	1.71	0.638	0.370	1.44	0 678	0.384	0 455
4	2.28	0.640	0.362	1.84	0.686	0.375	1 56	0.720	0.385	0.431
5	2.35	0.681	0.348	1.86	0.730	0.362	1.60	0.760	0.368	0.392

approximately. The quantity $C = \frac{M}{f_c' b d^2}$ depends on p and increases asymptotically to 0.455. This result is at variance with that obtained by Mr. Whitney, as it seems to show that for over-reinforced beams the ultimate resistance does increase with the steel ratio, although the rate of increase is slow.

It should be reemphasized that the numerical results cannot be more exact than the stress-strain curve from which they are derived.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

EXPANSION OF CONCRETE THROUGH REACTION BETWEEN CEMENT AND AGGREGATE

Discussion

By R. W. Carlson, Assoc. M. Am. Soc. C. E.

R. W. Carlson,[7] Assoc. M. Am. Soc. C. E. (by letter).[7a]—The disclosure of excessive expansion of concrete due to chemical reaction between cement and aggregate has opened a vast new field of concrete research. The importance may be tremendous, depending upon just how much bad concrete is attributable to this cause. A scattering of aggregates that are susceptible to reaction with alkalis doubtless will be found in all parts of the country.

If recurrence of cement-aggregate expansion can be prevented, the average quality of concrete structures should be raised appreciably. Recurrence of this type of concrete expansion probably can be prevented with least hardship to cement and aggregate producers when the following questions are answered:

1. Is the trouble due only to the formation of an alkali carbonate, only to the formation of an alkali silicate, or both?
2. Are the alkalis of sodium and potassium, which are usually lumped together in analysis, equally harmful?
3. Is it possible that the combination of moderate-alkali cements with the susceptible aggregates will produce expansion in correspondingly longer time?
4. Conversely, is it possible that the combination of high-alkali cements with moderately good aggregates will ultimately produce trouble?
5. Is the alkali carbonate likely to be formed by carbonation of the alkalis in aggregates, especially those like crushed feldspar containing a considerable amount of alkali?
6. Is it possible that concretes containing good aggregates but cements of high-alkali content are likely to develop alkali carbonates, and subsequently shrink excessively due to drying?

Note.—This paper by Thomas E. Stanton, M. Am. Soc. C. E., was published in December, 1940 *Proceedings*.

[7] Associate Prof., Civ. Eng., Mass. Inst. Tech., Cambridge, Mass.

[7a] Received by the Secretary January 16, 1941.

7. Are the alkalis in the cement equally harmful whether free or combined, either as sulfates, aluminates, or otherwise?

The great need, of course, is for an accelerated test to which questionable combinations of cement and aggregate can be subjected. Due to the slowness of the reaction, doubtful aggregates or cements cannot be proved soon enough merely by making up common specimens and observing their expansion. In some cases, the potential expansion might not develop for a year or more. Therefore, some sort of reliable accelerated test needs to be developed that will give the result in a shorter time. At present, it is indicated that certain sizes of aggregate are more harmful than others, also that certain proportions of cement to aggregate are more sensitive. Undoubtedly, temperature is also a factor. Some combination of conditions may be found that will give an indication of the long-time expansion in a few weeks. It is hardly to be expected that any type of autoclave test applied for only a few hours will be reliable.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FOUNDATION EXPERIENCES, TENNESSEE VALLEY AUTHORITY
A SYMPOSIUM

Discussion

By Messrs. James S. Lewis, Jr., Robert M. Ross, Verne Gongwer, Portland P. Fox, and James B. Hays

James S. Lewis, Jr.,[31] Assoc. M. Am. Soc. C. E. (by letter).[31a]—The writer would like to express his appreciation to those who contributed their opinions and criticisms of the material presented relative to the foundation experiences of the TVA and to acknowledge the great value of the information and ideas contained in the discussions of the Symposium. It is probably worth remarking that the authors of the Symposium papers, and the discussers, seem to be in general agreement upon certain features that could well be re-emphasized. For instance, the inadequacy of small core borings for obtaining comprehensive information about foundation conditions is recognized and the use of large core holes, which permit visual inspection of the undisturbed foundation, is becoming more common for exploration purposes. The economy of cofferdam grouting with cheap grout before unwatering, if the need is indicated, will probably not be questioned by one who has experienced the excessive operating and pumping costs attendant upon the maintenance of a wet cofferdam. The discouragingly difficult and expensive task of stopping foundation leaks, which may appear after water has been impounded, justifies the use of every reasonable precaution to prevent them.

Mr. Moneymaker's lucid description of the geology of the various foundations described in the Symposium adds materially to the information presented by the authors in describing the problems peculiar to the different sites. There is a growing inclination on the part of engineers to take greater advantage of geologic skill in connection with the construction of large dams, and the value of having a resident geologist attached to a project throughout the duration of

Note.—This Symposium was published in March, 1940, *Proceedings*. Discussion on this Symposium has appeared in *Proceedings*, as follows: May, 1940, by Messrs. George K. Leonard, and F. B. Marsh; June, 1940, by Messrs. Berlen C. Moneymaker, R. F. Walter, William F. Prouty, Jacob Feld, and A. Warren Simonds; September, 1940, by Messrs. V. L. Minear, and C. E. Blee; and January, 1941, by Barton M. Jones, M. Am. Soc. C. E.

[31] Constr. Supt., Watts Bar Dam & Steam Plant, TVA, Watts Bar Dam, Tenn.

[31a] Received by the Secretary November 27, 1940.

construction operations is recognized in the TVA. The opinions of the geologists have proved especially valuable when questions involving the necessity of removing rock of dubious quality have arisen, and sizable economies have resulted from the immediate availability of expert geologic opinion. It seems unnecessary to state that, ordinarily, every yard of rock excavation avoided is also a yard of concrete saved.

As Mr. Walter declares, no two dam foundations are alike, but the same fundamentals of successful treatment are frequently found to be applicable to widely separated sites. Although the methods used at Alcova Dam and at Norris Dam differed as to details, the problems encountered and the solutions found were similar in many respects. The successful use of fine dune sand as a medium of reducing costs when large openings were to be filled is another example of the manner by which economies may be achieved with materials available at comparatively little cost. The use of this material also doubtless reduced shrinkage as compared with neat cement grout and, as a result, formed a superior seal.

The wisdom of increasing the spacing of holes, when admixtures that retard the setting time of the grout are used, is questionable. Narrow channels, such as result from the weathering of vertical joints, may well be missed by the drill holes if they are not spaced at reasonably short intervals, and the writer feels it unwise to increase the uncertainties necessarily attendant upon work of this nature. The following costs, which were incurred at Norris Dam, are included at the suggestion of Mr. Feld:

Drilling, foundations	$ 234,286.75
Drilling, reservoir rim	138,018.13
Grouting, foundations	309,235.10
Grouting, reservoir rim	138,091.27
Excavating and filling tunnels	124,013.34
Total cost of all treatment	$ 943,644.59
Total direct construction cost of Norris Dam	$18,584,745.00

As Mr. Simonds states, it is essential that clay-filled seams be washed free of all loose material before grout is injected into them. In tunneling to seal an 18-in. clay-filled seam, a section of which had been grouted, under the east abutment of Norris Dam, an excellent opportunity was afforded to observe the action of grout in clay. This seam had not been washed, and excavation revealed that the grout had worked its way into the clay in fins and layers, filling any openings that might have existed and definitely consolidating and increasing the density of the material generally. However, despite the noticeable improvement, the seal was not positive and the condition was not such as to resist percolation safely and to resist possible subsequent erosion for an indefinite time. As Mr. Simonds indicates, engineers are frequently at variance in choosing between stage grouting and packer grouting. Both methods possess advantages as well as disadvantages. When time is limited, the use of packers tends to expedite the operations. Also, the expense of

cleaning out or re-drilling holes between stages is eliminated. Stage grouting, as a result of the repetitive process, should tend to form a tighter seal by filling any contraction spaces that may exist around grout of previous injections.

The questions raised by Mr. Minear in his interesting discussion might profitably be asked in advance of every program of foundation treatment—and the answers would probably be different for every site. These questions deal with the fundamental problems common to the grouting of all types of foundations, and the correct answers supply the key to satisfactory results and economical costs.

The determination of the proper pressure of injection usually depends somewhat upon experimentation, and it is generally safe to say that a horizontally stratified rock should be given all that it will stand without serious upheaval. The tiltmeter, which Mr. Minear describes, would certainly have decided advantages over the comparatively expensive installation of upheaval gages used at Norris Dam. However, the writer questions whether it would not be possible for simple vertical movement to take place without registering on the tiltmeter. Angular displacement would occur inevitably if movement continued, but it seems possible that damage might result before the tiltmeter registered.

The writer feels that fairly small holes may be used successfully for grouting when it is unnecessary to wash loose material from underlying seams. Holes of large diameter make it possible to insert devices for washing individual seams but, for grouting purposes, they probably possess no advantage over small holes. The higher fluid velocities that would exist in the smaller holes at any given injection rate would tend to keep them scoured clean and to prevent plugging in the hole. Mr. Minear's reasoning regarding the relation between the cost of drilling and the spacing of grout holes is logical and clearly stated.

Concrete-filled tunnels probably form a more positive seal against percolation than any other practical means that might be employed; and where seams are large and extensive, the cost of tunneling, which may be estimated with fair accuracy, may compare favorably with the cost of grouting. As Mr. Jones states, the tunnels at Norris Dam were located so that the seams which were followed formed the roof, and he indicates that this location was chosen in order to avoid disturbing the overlying strata of rock. It is extremely difficult to fill a tunnel to the top, completely, with concrete. It becomes necessary, therefore, to grout the roof of the tunnel after it is filled and quantities of grout may escape laterally to the seam, the desirability of which may be determined by the circumstances. If conditions do not warrant grouting the seam, in addition to sealing it with a tunnel, a better condition will result from locating the tunnel so that the seam forms the invert. When the tunnel is filled, the seam will be sealed completely, and any space remaining between the concrete and the roof of the tunnel may be filled with a relatively small quantity of grout.

The upheaval gage used at Norris Dam was designed by Mr. Jones and served its purpose very effectively.

ROBERT M. ROSS,[32] ESQ. (by letter).[32a]—When the paper was written, the water had not yet been impounded behind Guntersville Dam. Now that the lake has been filled for nearly two years, it may be of interest to review the resulting changes in ground-water conditions, in the flood plains downstream from the embankments, and to note the effectiveness of the cutoffs and grouting in the foundation and of the bulkhead in the nearby cave.

Leakage under the dam and around the abutments has been negligible. The total flow from the well points located in the flood plains, downstream from the embankments, may be considered of no engineering or economic consequence. Nevertheless, the behavior of the ground water in the flood plains is of considerable interest.

Before construction was started at Guntersville Dam, the area at the foot of the north abutment and downstream from it was damp and boggy during a large part of the year and numerous wet weather springs flowed during rainy periods. When the construction plant was built, in 1936, drains were installed and much of the low, boggy land was filled in with earth, rock, and gravel. According to foremen who supervised this work, the fill was sometimes as much as 6 or 8 ft thick. However, even with a system of drains, well point 7, located in this area, 275 ft below the axis, at station 3 plus 10 South, 2 plus 75 West, would flow after heavy or prolonged rains.

In the latter part of January, 1939, after the Guntersville Reservoir was filled, well point 7 began to flow and two small seeps appeared near it, at station 3 plus 8 South, 2 plus 78 West. Several wet, soft places and very small seeps also developed near the northwest corner of the parking lot and about the Time Office. The total flow was between 10 and 15 gal per min, at first. It diminished later, but varied between 2 and 20 gal during February, showing increases after rains. No large increases occurred until the night of March 29 when, after a very heavy rain, a seep delivering about 22 gal per min broke out in the road. It continued to flow until it was diverted through an underground drain. Since March 29, 1939, a flow of between 15 and 30 gal per min has been maintained. In March, a number of additional well points were drilled so that, since then, very detailed information on the ground-water levels has been available. The water flowed from several of these well points until extra sections of casing were added to them. Well point 7-E, located about 2 ft from 7, took most of the water from the latter.

Four methods of determining whether the water coming from the well points and seeps was ground water percolating down from the abutment and the plateau remnant back of it or water leaking from the reservoir were followed. These were (1) a study of ground-water elevations and seepage as related to rainfall and reservoir levels, (2) tracing subsurface water movements by means of fluorescein, (3) chemical analyses, and (4) comparative temperature readings from the water in the well points and in the reservoir.

Lack of space makes it impossible to describe these investigations and their results in detail. It must suffice to say that the ground-water studies, chemical analyses, and temperature readings all indicate unequivocally that the water

[32] Asst. Geologist, TVA, Water Control Planning Dept. Geologic Div., Paris, Tenn.

[32a] Received by the Secretary January 16, 1941.

coming from well points 7 and 7-E and the nearby seeps is ground water. The experiments with fluorescein, having had only negative results, are inconclusive.

Part of the flow from the well points is undoubtedly ground water and, in the writer's opinion, it all is. Nevertheless, before the filling of the reservoir, seepage occurred only intermittently and the water in the abutment stood considerably lower than it has since. The levels in the well points on the abutment only reached an elevation of as much as 585 during rare periods of very heavy rain, usually standing 10 or 15 ft lower, while, since January, 1939, they have never gone below 585. Obviously, a marked change has occurred and the reservoir is responsible. The explanation is probably this: Before the gates of the dam were closed, the ground water in the large plateau remnant against which the dam abuts seeped away to the river underground, passing through a large area both upstream and downstream from the axis. When the reservoir was filled, the outflow area was sharply reduced and the flow of ground water concentrated downstream from the abutment. This upset the previously established equilibrium of flow and produced a considerable rise of the water table in the abutment. The additional hydrostatic head on the outflowing water beneath the foot of the abutment caused it to well up to the surface and issue from seeps and well points.

Mr. Gongwer has given a comprehensive account of the ground-water behavior in the south flood plain and the thorough and detailed investigations that were made there. His paper covers the history of the ground-water investigations up to the fall following the filling of the reservoir and, in his closing discussion, he has brought the history up to date. For these reasons, any account of the studies on the south flood plain, by the writer, would be superfluous. These studies were made under the direction of Mr. Gongwer and many of the methods used were originated by him. They included investigations of ground-water levels as related to rainfall and reservoir elevations, extensive drilling, sounding, and contouring of the water table, measurement of flows from flowing wells, the use of fluorescein and geophysical apparatus, chemical analyses, comparative temperature readings, and correlation of water levels with barometric pressures. The studies yielded much valuable and exact information which, the writer believes, points to the following conclusions.

There is an inflow into the area, downstream from the central part of the south embankment, of something more than 140 gal per min. Drilling and ground-water studies indicate the flow to be rather deep, through seams in the foundation rock, rather than through the overlying alluvium. The seams in the rock and the permeable portions of the alluvium are all interconnected, however, as closure of the valves of the flowing holes produces a quick rise in practically all the holes in the vicinity. The underground reservoir must be comparatively tight and of limited size; otherwise, hydrostatic pressure could not be built up so rapidly in the holes. Since a flow of 140 gal per min reduces water levels as much as 7 ft, the inflow must be comparatively small. Before the holes were drilled which tapped the water beneath the flood plain, the water must have accumulated slowly, under pressure, being held down by the cover of impervious clay that blankets the flood plain. Its outlets were ob-

viously much restricted, so that instead of moving away freely it rose high in the alluvium.

The most obvious source of the water is, of course, the reservoir, although the possibility that it is derived from the highland beyond the abutment has also been considered. A divergence between its chemical composition and that of the reservoir water lends weight to the belief that it may be ground water. However, the dificulties involved in the transmission of ground water from the abutment to the center of the flood plain, under sufficient pressure to cause it to flow out at the surface, appear so great as practically to preclude the abutment area as a source. Ground water drains away most freely near the abutment and stands considerably lower there than in the area of flowing wells. This condition could hardly exist if water were moving from the abutment to the wells.

In the writer's opinion, the reservoir is entirely responsible for the flow, even though analyses of the water reveal it to be more similar to ground water than to reservoir water. It must be remembered that the alluvium of the flood plain contained a large volume of water previous to the filling of the reservoir. When the reservoir was filled, pressure was put on this store of ground water so that it was forced through some small crevice or seam under the axis of the dam, in the foundation rock, which had, somehow, escaped complete closure. The water accumulated under pressure, downstream from the dam, until it was tapped by drill holes and flowed from the surface. Since the flow is relatively small, the ground water being pushed ahead of the reservoir water has not yet been exhausted and may not be for some time. If this hypothesis is correct, the contents of the water flowing from the holes will gradually approach that of the reservoir water and will ultimately approximate it.

The flow downstream from the dam is negligible and, by allowing the water to drain freely, ground-water levels may be held below the surface of the ground at all points. The south embankment may be considered unusually tight and effective. There is no likelihood of piping occurring below it.

Stream flows in the valley below the dam, in which the inlets to the large cave are located, have been carefully checked, and thorough observations of inlets and possible areas of seepage made. These indicate that no leakage whatever is occurring through the cave, which was blocked with a concrete bulkhead as a precautionary measure.

Mr. Marsh has pointed out the danger of gradually increasing leakage through limestone foundations as a result of progressive enlargement of seams and channels in the rock. The development of such leaks has presented a difficult problem at numerous dams. The Great Falls Dam, on the Caney Fork River, in Middle Tennessee, is a notable example. When the dam was built, in 1916, and for a good many years thereafter, leakage was relatively insignificant. As time passed, seepage through the narrow limestone divide that separated the reservoir from the river valley below the dam became steadily greater. The leaks increased in size and many new ones appeared. At present, water flows freely from a series of openings that extend from the dam to a point nearly a mile downstream from it.

However, careful studies indicate that the augmented flow has been produced by the washing out of clay filling which almost plugged an extensive system of previously existing solution channels, rather than by any further solution of the limestone itself. Numerous channels existed, but they had been formed by the extremely slow solvent action of water, operating over hundreds of thousands of years. It is interesting to note that now, since most of the clay has been washed out of these channels, the rate of increase of the leakage has markedly diminished.

At every dam with which the writer is familiar, where enlargement of openings in a limestone foundation has occurred, such enlargement seems to have been caused by the washing out of an accumulated filling rather than by actual solution. Although a well-consolidated limestone is a soluble rock, in comparison to most other rocks, it still is likely that many thousands of years must elapse before it would be affected appreciably by solution. If all seams and openings in a limestone foundation are thoroughly cleaned, so that all clay, sand, or other unconsolidated material which may be present is effectively removed and the cavities, consequently, can be completely blocked with grout or concrete, no troublesome leaks should develop from subsequent solution of the foundation during the life of the dam.

The writer appreciates the interest of those who have taken part in the discussion of the paper. Their comments have brought out various pertinent features relative to limestone foundations.

VERNE GONGWER,[33] M. AM. SOC. C. E. (by letter).[33a]—To those who have taken occasion to discuss this paper, the writer wishes to extend his thanks. Mr. Leonard corroborates the fact that the grouting under the second-stage and third-stage cofferdams was greatly advantageous to speedy and economical prosecution of the work. Had the work in the spillway (or second stage) cofferdam not been performed in record time during the summer and fall of 1937, the final operation of the generating units, and to that extent some phases of national preparedness, would have been impeded.

The writer, however, disagrees somewhat with Mr. Leonard as to the Guntersville Dam site being a "supersite." It can be agreed that probably "Few dam foundations in limestone are found that are better than that under Guntersville Dam," if taken in the literal sense, since nothing but very sound limestone and comparatively few uneradicable seams and caverns, solidly and tightly filled with good concrete or grout, were left under any part of the dam or any of its appurtenances. The large amount of inferior and undermined rock which it was necessary to excavate to make this a reality, plus some good rock necessarily excavated for the deep draft tubes, less an approximately equal quantity wasted at various points, was sufficient to riprap completely the 35-ft high river banks for about one-half mile downstream from the dam on both sides of the river, and to a thickness of 6 ft to more than 12 ft.

Where Mr. Leonard states "In the entire length of 4,000 ft, only two areas totaling about 600 ft in length showed signs of being cavernous or unsound;

[33] Tacoma, Wash.

[33a] Received by the Secretary January 6, 1941.

and these areas were fairly close to the surface," he possibly refers only to the length of two peculiarly cavernous areas along the axis under the south earth dike from station 31 + 65 to 37 + 10 and from station 39 + 50 to 40 + 15, totaling 610 ft (see Fig. 49). After costly deep excavation between double rows of interlocking steel sheet piling, these areas were found to be totally undermined and very cavernous, with partly filled seams of considerable lateral extent defying exploration. They occurred as high as 12 ft, directly under the line of the cutoff wall, with many branching arms and broad open seams, ramifying both upstream and downstream. It cannot be quite agreed that they were "fairly close to the surface," or at least not in the sense that they could be easily reached and corrected, since some of them lay as much as 25 ft below the top of solid rock and 60 to 65 ft below the ground surface.

In addition to the 610 ft of major caverns in this one local group, found upon opening up the south cutoff, Mr. Leonard has apparently overlooked other

Fig. 65.—Cavernous and Underground Solution Seams Encountered in Excavating for a Sump in the Lock Cofferdam

definitely cavernous areas—(*a*) 200 ft along the north cutoff foundation (see Figs. 30 and 32); (*b*) approximately 320 ft along the axis from the center line of the lock to and including pier No. 6 of the spillway (see Figs. 33, 34, 36, and 65); (*c*) 965 ft from station 22 + 00 to station 31 + 65, most of which required more or less trenching into the rock, and where water flowed copiously from seams and grout holes, also fluorescein from several hundred feet away; (*d*) approximately 200 ft under the mountain slope, constituting the south abutment, requiring much deep drilling and grouting to insure watertightness of mud seams; (*e*) small water-bearing caves leading into the sump common to the lock and spillway cofferdams (see Fig. 65); (*f*) an extensive undermined area with solutionized horizontal and vertical joints beneath and downstream from the north one third of the spillway apron where all exposed rock was

dowelled to the sound rock below the shale, filled and grouted; and (*g*) small caverns near the junction of the tailrace wall with the southwest corner of the power house. Exclusive of the south abutment, the foregoing shows 2,100 ft along the axis of the total 4,000 ft-length of the dam, as cavernous and undermined, necessitating major correction. Certain of the other items caused flooded cofferdams, excess equipment breakage, and much delay and expense.

Many long, deep, vertical solution channels in the remainder of the total length of the river-bed section, embracing most of the spillway and power

FIG. 66.—VERTICAL SOLUTION SEAMS UNDER INTAKE AND POWER HOUSE, CLEANED AND NEARLY READY FOR CONCRETE FILLING

house, required painstaking and expensive remedial measures for opening, cleaning, and filling, of which Fig. 66 is typical. In addition, it was considered necessary to drill one or more 36-in. holes on each vertical seam and fill with concrete for blocking the seams. Also in the two 36-in. holes drilled and concreted under each spillway pier, the designing engineers required embedment of heavy vertical steel and column hoops to increase shearing resistance of the shale seam. Fig. 38 is misleading; it was selected expressly to show one local perfectly sound area where typical river-bed erosion (as distinguished from solutionizing) could be seen clearly. Practically all other foundation areas in all three cofferdams required more or less extensive correction or excavation to the general shale seam. At a conservative estimate, a concrete cutoff to shale, for the 2,100 ft of definitely seamy and cavernous rock, would have cost about $350,000, and the disturbance of adjacent rock due to the blasting in the trench and other indirect and consequential costs could easily have doubled or tripled that sum.

As to whether several other sites could have been developed, feasibly or economically, near the head of Wheeler Pool, reference is made to the paper by Mr. Ross (see heading "Selection of the Site"), in which the history of the search

for a dam site in this particular reach of the stream is summarized. From the record related it would appear that the "Coles Bend Bar" site (later known as the "Guntersville" site) was practically a "ground hog case," there being no other feasible site found after diligent surveys and explorations.

Eight sites were investigated at considerable cost, seven of which were eliminated as totally unsuitable. There evidently was no other economically or physically feasible site within the limits of the reach of the river available for building a dam to fit properly between the newly completed Wheeler Dam and the older Hales Bar Dam, approximately 80 miles upstream, completed about twenty years previously (see Fig. 22). It is doubtful whether the locating engineers fully realized either the many disadvantages of the site or the few redeeming features which later developed.

Mr. Leonard is probably correct in the assumption that the shale seam outcrops some distance below the dam, but there was no direct evidence to that effect. On the other hand, it is quite possible that the dip may slightly reverse shortly in that direction. Whether or not this shale seam does outcrop below the dam, subsequent contract dredging operations for channel improvement below the dam have revealed very extensive and troublesome deposits of massive residual boulders, produced by the processes of solution, such as are seen in Figs. 27, 28, and 42 and actually excavated; also at least one deep solution channel more than 50 ft below the stream bed was discovered. It is not believed, therefore, that a location in this area would have been advantageous. Also, there was nothing within the next 4 miles high enough or near enough to form a right abutment for the dam.

Honeycomb Cave did not add to the desirability of the Guntersville site (see Fig. 26). This cave, with upper and lower passages and multiple mouths considerably below pool level, was surveyed for approximately one-half mile and traced by fluorescein about 2 miles farther to a number of sinkholes downstream from the dam. Several of these sinks had earth rims slightly above pool level and bottoms in solutionized rock somewhat below pool level. Direct connection from the unsurveyed part to the river downstream from the dam was a distinct possibility.

The only feasible place where this intricately branching cave could be blocked off successfully was finally located about 700 ft inside of the mouth. The consultants and TVA engineering heads were considerably concerned with regard to the possibility of other thinly masked outlets, branches, or connections under the valley floor, or possible connection with one of the several springs below the 595-ft contour. Upon completion of the plug (at considerable cost), the cave was filled rapidly, by the water of the small creek therein, to the floor of the upper passage, 6 ft above pool level; and during approximately two years no leaks were discovered below the dam. Had such potential leaks occurred, and had no good place been found to plug the cave in advance, it would have been very costly and possibly very unfortunate, as construction of the dam was already well advanced. Great credit for the careful survey of this cave and the location of the only possible point for blocking is due Mr. Ross, and possibly others of whose participation the writer may not be aware.

The foregoing should not be construed as disparaging the preliminary and planning work of the engineers of either the Army or the TVA, or of Mr. Leonard's valued opinions or able discussion. It is intended merely to show that Guntersville was a definitely difficult site and to call attention to the narrow margin of feasibility, and necessity for grasping every favorable feature, involved in planning and constructing a chain of dams on such a river as the Tennessee—all of which redounds to the credit of the TVA engineering organization, in general.

As stated by Mr. Leonard, it is doubtful, in any case, if deep grouting below the shale seams could be dispensed with properly, in that, despite all the care and expense incurred, there yet remains in some minds a question as to whether or not the source of the water, producing the pressure phenomena in the rock seams below the dam, does not come from uncorrected channels connected with the reservoir. In this connection, the writer has recently been apprized (without opportunity to gather or verify further data) of the existence of a theory, approaching fact, that there is artesian pressure under the entire Tennessee Valley floor. In that case, the "ground-water hump" phenomena would be readily explained. It is regretted that a few very deep holes could not have been drilled within a mile or so downstream to check this theory.

Certain of the grouting and cavern filling methods are believed to be actually unique and more or less original, such as (1) the drilling and filling with concrete of "interlocking" 36-in. holes; (2) the filling of horizontal seams of some height with fine-aggregate concrete through drill casings driven into them from the original ground surface; (3) the concreting of other seams through 36-in. drill holes, between forms consisting of tiers of concrete-filled and sand-filled gunny sacks; (4) the jetting of overburden into caverns and "lost" ground downstream of the cutoff in such a manner as to balance the pressure of puddled backfilling between the two rows of steel sheet piling; and (5) the successful and advantageous sealing of cellular steel cofferdams by grouting of underlying open seams in limestone or other material by drilling down through the center of the cells into the seams or between and under residual boulders.

Notwithstanding an early report of the board of consultants to the general effect that cutoff provisions possibly would not be necessary at Guntersville Dam, the positive need for them was later recognized and authorized by the responsible engineering heads. During an early inspection were viewed: (*a*) The terrane, test pits, and spoil; (*b*) compartment boxes of samples of silt, sand, and gravel obtained by sinking 24-in. casings to rock where ground water had defeated open-pit methods; (*c*) the logs of tests with a small pump on a line of test pits from the river bank to the south bluff; and, more particularly, (*d*) the record of one of the pits not far from the south bluff where the pump (125 gal per min) could lower the water only 0.2 ft per hr. At the latter point sand ran under the sheathing of the pit and was pumped out copiously. It is probable that these data were not available at the time of the report mentioned.

It is worthy of note that this pit was adjacent to the area where the most serious and but partly filled seams and caverns were later encountered and required major corrective measures. The sample boxes for this pit, and an

adjacent pit to the south, revealed materials close to bedrock ranging from fairly clean sand and gravel to large cobbles. Significantly, a characteristic valley in the ground-water contours now exists in this vicinity. These materials and many definite iron-stained open water courses therein were revealed in the excavation along the cutoff piling. As to unfavorable conditions to be subsequently encountered below the surface of the rock itself, none of the preliminary borings of the U. S. Engineers or of the TVA, on the customary rather wide spacing, gave more than slight indications, and these indications were interpretable only after detailed explorations and construction were well advanced. That these conditions were serious, and that they required positive and expensive remedial measures, with results equivalent to those adopted, is obvious. Apparent failure to recognize or to cope adequately with assumedly similar rock conditions at the old Hales Bar site resulted in extensive subsequent leakage. Had not the seriousness of the conditions under the flood plains at Guntersville been recognized, and had not the extensive and expensive measures been adopted which were used, it is the considered opinion of the writer, concurred in by a number of experienced engineers familiar with the site, that piping and underflow would certainly have resulted, under very similar conditions (but on a much larger scale and with much more serious results) to that which occurred at a smaller dam, near Reading, Pa.[34]

The great extent of large residual boulders was evident only after the sheet piling had been driven and after the irregular seating of the piling had been investigated by exploratory excavation. The findings necessitated an arduous program, important both as to first cost and as to effect on construction schedule and indirect cost.

One of the main points that the writer desired to emphasize was that, in such erratic and temperamental terrane as soluble limestone (and, in fact, at most proposed dam sites of any nature), preliminary investigations, while avoiding, if possible, unnecessary expense, should go beyond bare customary practice and should be sufficiently extensive to obviate such surprizes as were experienced, and such changes of plans as were necessitated, at Guntersville Dam. Preliminary investigations of this scope should contemplate, and include from the outset, sufficiently complete and effective methods, and sufficiently heavy-duty pumps, drills, and other equipment, to obtain the necessary and vital results and data. The corrective measures necessitated meticulous attention. As construction engineer, Mr. Leonard is to be highly commended for the final satisfactory and watertight results. The writer agrees almost entirely with the remainder of Mr. Leonard's discussion and wishes to thank him for submitting it.

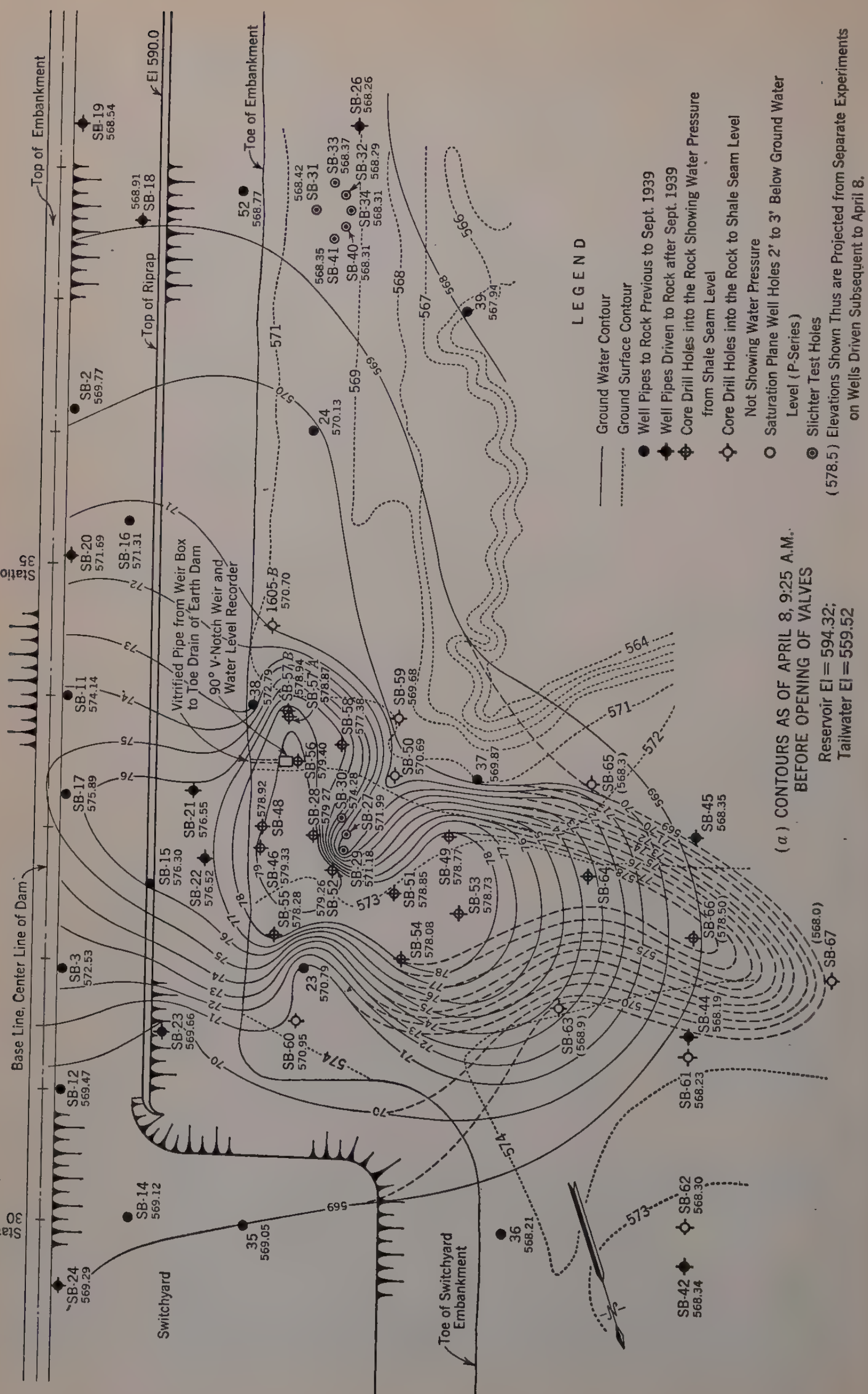
Although the writer is of the opinion that limestone seams "solutionize" very, very slowly (requiring perhaps 100 to 500 years for any appreciable or dangerous enlargement to develop), the remarks of Mr. Marsh are particularly well taken. In principle, they are definitely pertinent to any dam site, in any formation or kind of rock, and to all other engineering structures as well.

[34] "Reading Starts Work on New Water Supply System," *Engineering News-Record*, Vol. 113, 1934, p. 323; "Unmapped Eroded Limestone Complicates Corewall Job," by Farley Gannett, M. Am. Soc. C. E., *loc. cit.*, Vol. 115, 1935, p. 328; and "Caverns Under Dam Corewall Set a Nice Repair Problem," by Farley Gannett, *loc. cit.*, Vol. 116, 1936, p. 492—see also p. 502.

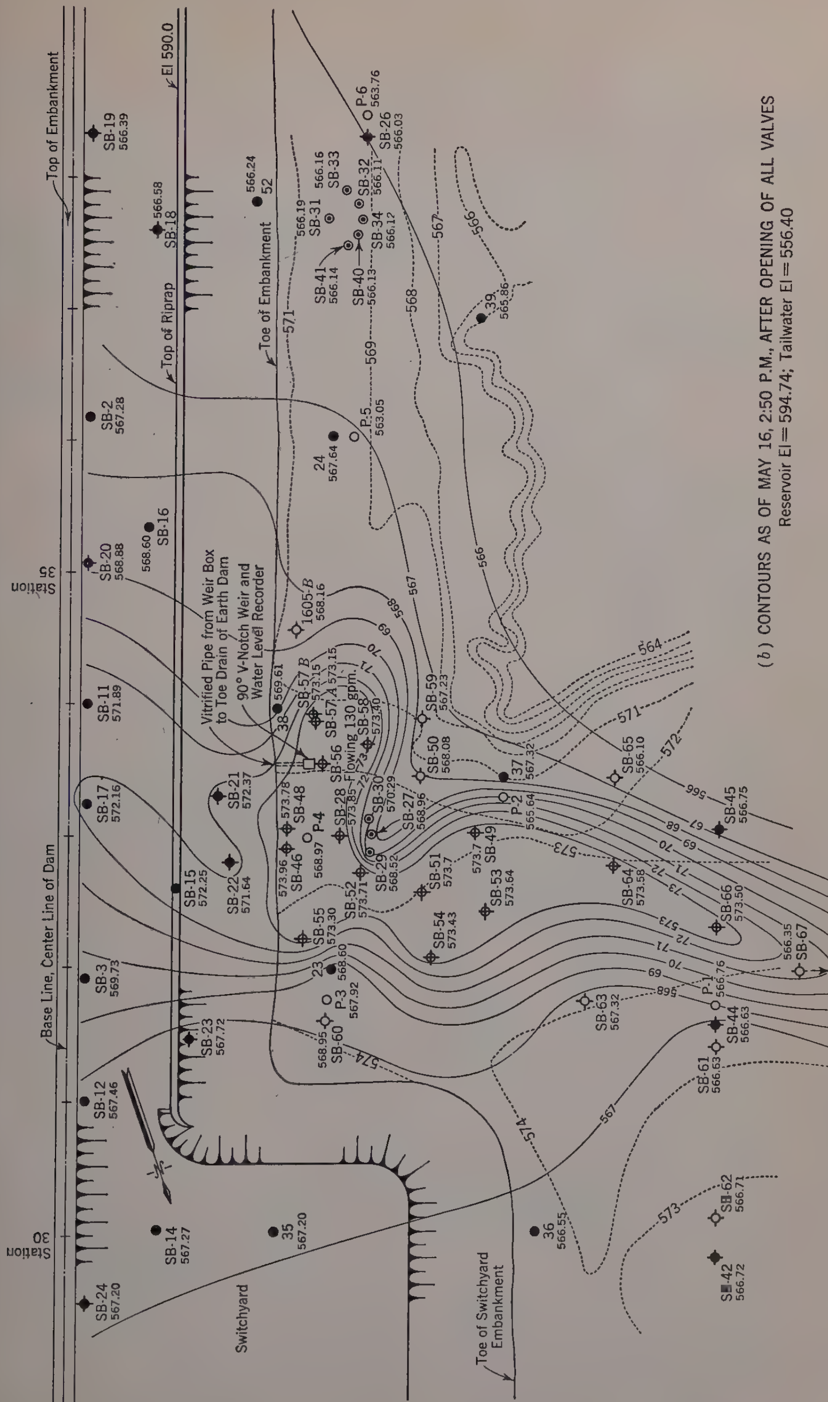
In continuing the observations of the ground-water wells, as routine procedure, and for the purpose of clearing up some uncertainties in properly depicting the ground-water contours, several additional observation wells were driven after the paper was presented. It can be readily seen and understood that the problem of locating wells for maximum information is a process of trial and error, and that always some previously driven wells might well be dispensed with and a few additional wells could yet be desired. Also, at the end it is usually regretted that the final nearly ideal system of wells could not have been foreseen and driven in the beginning so that a long, complete record could be had. Frequently, also, early conceptions of the behavior of the ground water are considerably revised as the study progresses. It is practicable in the light of the later findings and observations to return to a review of the early days of the work (when fewer wells, less satisfactorily located, existed) and, with care, reinterpolate and reinterpret the early ground-water maps, thus leaving a more understandable record and gaining a much better idea of the conditions than, likely, was held during the progress of the work. This was not done on this job for lack of time, and since the results as to underseepage were satisfactory.

If possible, it was desired to delimit the boundaries of the pressure zone or chamber in the rock previously discussed. To these ends, several additional well pipes were sunk to the rock and seated therein; holes were cored into the rock with 3-in. shot drills to, or below, the general shale seam. In order to measure any artesian water encountered, and to keep it from escaping into the gravel, without the use of "packers," the perforation of the bottom sections of the casings was omitted, and the casings were seated tightly in the rock before core drilling was begun.

To obtain the true "saturation" or "free ground-water" level, several short pipes (perforated near the bottom) were driven to a depth of only 2 or 3 ft below the point where standing water was encountered. The location of most of these (designated as P-1, P-2, etc.), as well as of the wells cored into the rock, is shown on the ground-water maps for 1940 (see Fig. 67). Except for the new holes in the "pressure bump," comparison of the ground-water contours as of March 26,1939 (shown in Fig. 45), with those in Fig. 67 demonstrates that in the approximate one-year interval the ground-water levels, in general, have been practically stationary. Only slight variations were produced by precipitation, possibly to some extent by tailwater, and more remotely, if at all, by headwater variations. In fact, a slight general lowering of the ground-water or pressure table is evident. The foregoing is also evident by a comparison of the water levels shown for March 31, 1939, in Fig. 46, with those for the same holes as of March 26, 1940, shown in Fig. 68. In Fig. 68, closely grouped curves are omitted after April 2, and the holes plotted were chosen as representative of a group. Straight lines represent holes read at intervals. Continuous graphs of wells SB-22, SB-52, and SB-15 are taken from recording instrument records. No wet or soft spots or new seepages have appeared below the dam during this period. Contours of the position of the "free" water table or saturation plane have not been attempted from the few shallow wells, but com-



(*a*) CONTOURS AS OF APRIL 8, 9:25 A.M. BEFORE OPENING OF VALVES
Reservoir El = 594.32; Tailwater El = 559.52



(*b*) CONTOURS AS OF MAY 16, 2:50 P.M., AFTER OPENING OF ALL VALVES
Reservoir El = 594.74; Tailwater El = 556.40

FIG. 67.—GROUND-WATER MAP FOR 1940, STATION 30 TO STATION 38, SOUTH EMBANKMENT

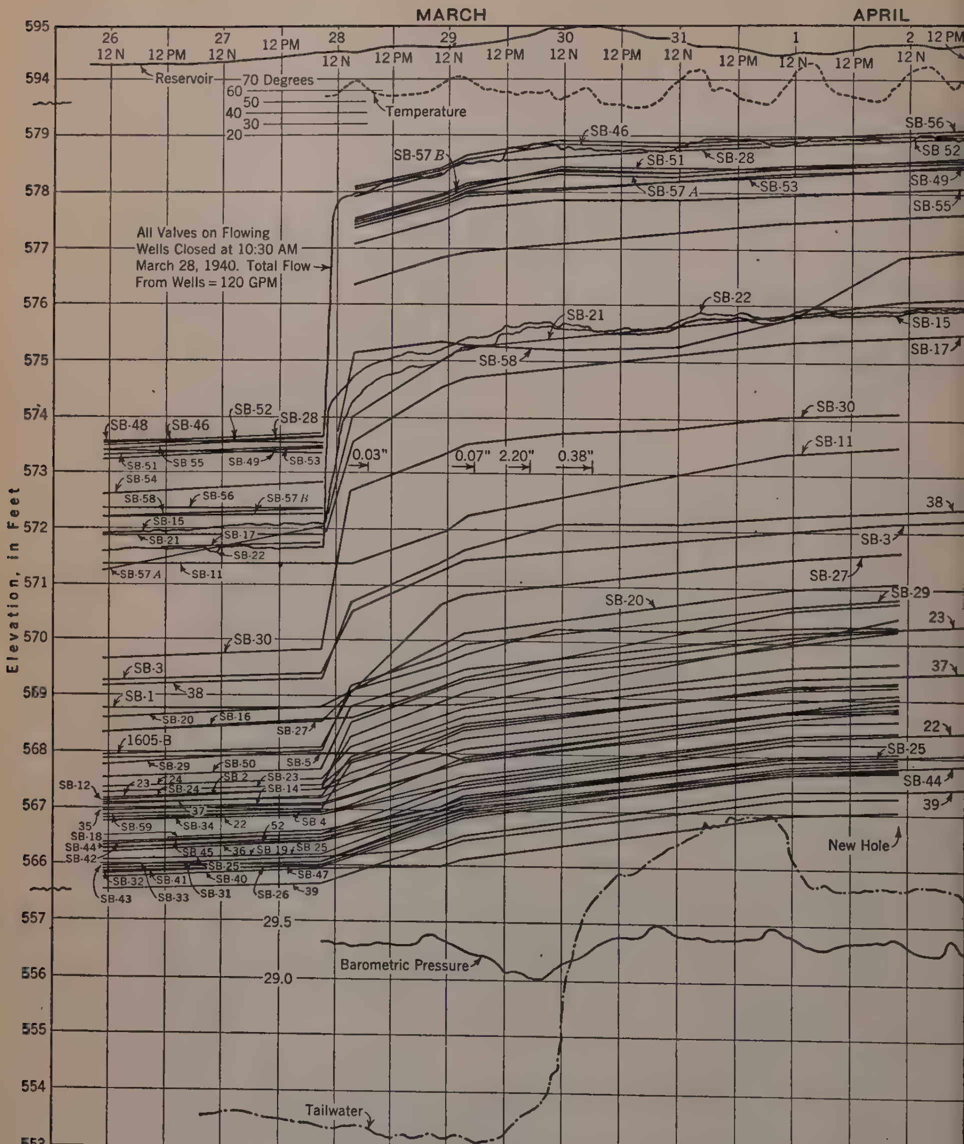
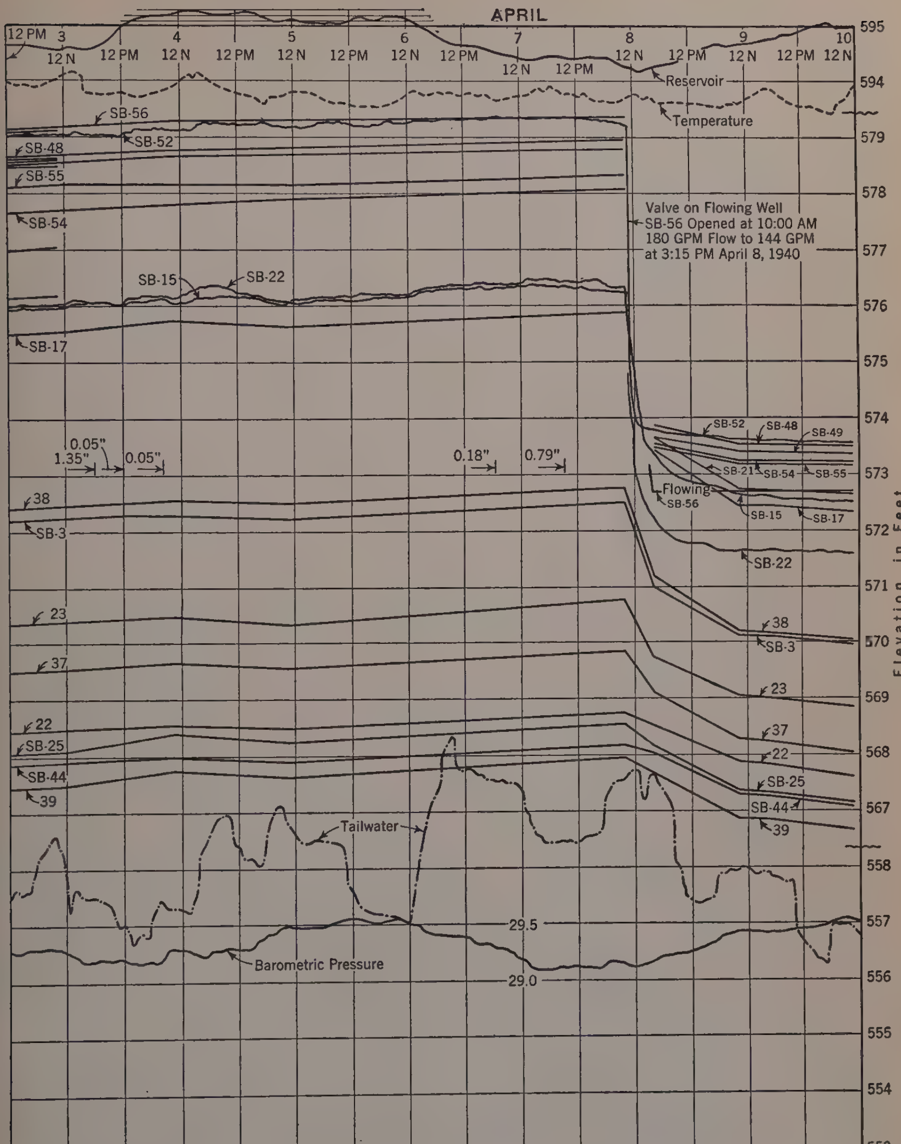


FIG. 68.—COMPARISON OF WATER LEVELS



in Wells During Pressure Experiment

parison of the elevations shown in Fig. 68 vary from fairly close agreement between shallow and deep wells (as between P-3 and SB-60) to a difference of about 7.3 ft (as between P-2 and SB-49). A rather exceptional coincidence exists under normal conditions between P-1, SB-44, and SB-61 (shallow, to rock, and into the rock, respectively); yet the last two respond positively when the flowing wells are shut off, as in the pressure experiment.

The various phenomena connected with the progressive driving of the additional wells were very interesting and helpful toward an understanding of conditions and would assist in future similar investigations. However, a detailed description is not possible for lack of space. The first flowing well resulted from drilling hole SB-28 down into the rock with the intent of applying a Slichter determination to the flow in any rock seam encountered. The most significant phenomenon was that as each succeeding flowing well was encountered an immediate lowering of the water level in well SB-22 resulted, the total effect being observable from the several ground-water maps. All wells as drilled were equipped with tees, valves, and static riser pipes, from manipulation of which some very helpful data were obtained. All flowing wells were extremely sensitive to back-pressure from well SB-56, or whatever previous well had the greatest flow.

Fig. 68 shows the graphs of the water surface in all wells existing at the time of the experiment; headwater and tailwater elevation; barometric pressure; temperature; and date and quantity of rainfall. The latter, and doubtless to some extent the high tailwater, caused by the necessity of spilling water from the reservoir, created a slow rising trend in all wells, beginning 24 hr to 36 hr after the intital rise due to closing of valves. This trend continued until the moment of opening the valve on SB-56. It should be noted that the substantial and fairly rapid variation in headwater is definitely not reflected in the behavior of the wells.

As indicated by the graphs of the three automatic recorders, the behavior of the three wells, SB-15, SB-22, and SB-52, was remarkably similar and evidently is representative of the manually observed wells wherein the detailed variations were impracticable of observation. The recorder graphs of wells SB-15 and SB-22, which were driven only to rock, agree exactly, whereas that of well SB-52, which was cored down into the seams in the rock, does not quite agree as to every detail of variation. Quite obviously, none of them shows headwater influence, and rainfall influence is but slightly indicated. However, several sharp rises in all three automatic graphs are coincident with sharp drops in barometric pressure; and, also, the reverse is noted, being in all cases sharply coincident as to time. The water stage recorders and barograph produced automatic records; and therefore no personal error or influence is involved. One or two longer trends between water levels in wells and barometric pressure appear to be likewise related. Sharp tailwater variations do not seem to affect the graphs.

Barometric influence on the water levels in the wells, thus definitely established, would appear to indicate substantial tightness of the underground pressure chamber or seams, definite freedom from any but the slightest reservoir influence, and largely predominant artesian or remote water supply rather than

any considerable supply from reservoir seepage even if it occurs through deep and circuitous channels. If any considerable part of the well flow were from the reservoir, the route would have to be long, very constricted, and circuitous; otherwise, barometric variations would be balanced between the reservoir surface and the water surface in the well casings and would produce no variation in the water level in the wells. Since pressure variations are transmitted practically instantaneously, even through long, very small, or capillary tubes, it would seem evident that the reservoir can have little more than capillary connection with the pressure area, making allowance for modified or dampened action in case the flowing water is supplied predominantly from artesian or other source. In the opinion of the writer, barometric influence is quite definitely indicated by the graphs, and comparison of barometric and water-level graphs therefore seems a very desirable and advantageous phase of such an investigation.

The writer's attention was called by Mr. Harris to the apparent influence of temperature variations on water levels in the wells. There appears to be some similarity between the temperature graph and the well graphs in a tendency to rise during the late forenoon and then to level off in the afternoon. On certain days, however, marked temperature rise is not accompanied by rise in the wells, and in certain of those instances the barometric influence seems to have been the controlling factor and to have been opposite in effect to temperature. It is probably true that a marked temperature rise, when the water is standing several feet above the ground in the pipes, would warm up the top of the water column and cause a slight rise in the water surface independent of reservoir level, which does not, from the graphs, seem to have any definite influence on the wells, either sharp or long range. However, the influence of the barometric pressure variations seems to be the stronger and quite sharp. No doubt some combination of the several influences affects the height of the water columns in the well risers. Temperature would operate independently of the source of the pressure water, whereas barometric pressure effects would be operative in discernible amount only where the source is comparatively remote, considerably constricted and circuitous, or artesian, and the pressure chamber substantially tight. All evidence indicates that the barometric influence is much the strongest of the several possible influences, and definitely indicative of predominantly artesian supply.

Fig. 67(*a*) shows the location of all wells in the study area existing at the beginning of the experiment, the water levels in these wells, and the corresponding highest ground-water (or pressure water) contours immediately prior to reopening of the valves. A "before map" preceding the experiment is not essential since it would be practically identical with the "after map," Fig. 67(*b*). In well SB-56 elevations were taken just before opening the valve at 10:00 a.m. In Fig. 67(*a*) holes with elevations in parentheses were drilled after this date. The water-pressure contours have been shown by solid lines as they were justifiably interpretable on that date (April 8) from the then existing holes; also by dotted lines as projected or extended by careful estimation from full field acquaintance to correspond with the certain behavior of wells SB-63, SB-64, SB-65, SB-66, and SB-67, driven subsequent to the experiment. In a study of

this kind, this illustrates strikingly the limitations and errors introduced by an inadequate number of exploratory holes. Probably additional wells near SB-67 would have shown that the pressure area extended still farther downstream (or westerly). It is believed that the zone was quite thoroughly defined in all other directions.

Fig. 67(*b*) shows the ground-water (or pressure) contours as of May 16 at 2:50 p.m., some days after the opening of all valves. The elapsed time interval was necessary to utilize the observations of the last and final wells driven at the site. Fig. 68 shows the wells practically returned to normal on April 10. In general the rise in all pipes during the experiment was at approximately the same ratio as in the preceding experiment (see Figs. 48 and 49). On April 8, upon opening of all valves at 10:00 a.m., well SB-56 discharged 180 gal per min. This dropped to 144 gal per min by 3:15 p.m. and shortly returned to normal (120 to 130 gal per min).

As "one well led to another" in searching out the boundaries of the pressure area, the size and shape of this area changed considerably from earlier impressions. It is noteworthy that the wells registering the highest normal water levels, with all valves open, lie in a comparatively straight line through well SB-56 and approximately station 35 + 00 of the dam axis. Well SB-67 was located with this in mind but failed to strike the solution channel or pressure seam, probably by feet or possibly only by inches. It also will be noted that wells Nos. 1605 B and 19 (fluorescein wells), shown in Fig. 45, indicate a drainage or seam axis of the same general direction, passing through approximately station 35 + 25. However, this latter seam or drainage system evidently was at a different level from the one previously discussed, and apparently not connected with it, as it will be noted in Figs. 45 and 67(*a*) that well 1605 B is not affected by pressure rise during the pressure experiments. Also it is evident that this seam was effectively blocked by the cutoff. The wells active during these experiments evidently pierced a seam or system of seams at a different elevation or horizon. This is understandable since many of the solution seams at different levels follow the general direction of one of the principal directional axes of jointing.

Numerous caverns with important arms and seams branching from them were observed in excavating along the cutoff trench, some of which could have been more deeply explored and mapped, at some possible risk, had the pressure tests and the questions raised by them been foreseen at the time. Certain of these branch caverns lead off in the general direction of wells SB-56, SB-28, SB-49, etc. It is advised, therefore, in similar investigations, that all such tributary seams and arms, together with vertical seams and chimneys rising from these seams, be surveyed and mapped as completely as possible wherever and whenever encountered.

All questions as to the integrity of the dam and cutoff provisions having been answered satisfactorily, at the substantial conclusion of construction of the dam and power house about May 1, 1940, arrangements were made for periodic future observations in all ground-water wells and saturation-line wells in the rolled-fill sections of the dam; and a permanent installation was devised and constructed for measuring the flow of the key well, SB-56, which comprised

the total known flow from the pressure zone. This installation consisted of a concrete box, with its top flush with the ground surface, comprising a stilling well with baffles and a V-notch weir, with vitrified pipe outlet discharging into the toe drain of the earth dam, and with an automatic water level recorder in the stilling well. All well pipes were cut off just below the surface, tamped about tightly with clay, to exclude possible free entry of rain water, and provided with screw caps for discouraging unauthorized tampering but permitting continued observation. None of these pipes would overflow upon uncapping, with well SB-56 flowing freely, and should they do so later, due to possible future changes, the observer, upon removing the cap, would lightly screw on a riser pipe of sufficient length to permit the necessary observations.

Before leaving the shallow "true" ground-water wells, and the saturation plane wells in earth embankments, all such holes were earth-augured for a foot or two below the bottom of the casing to insure free entry of ground water. It should be mentioned here that extreme care should be used to ascertain that all observation wells are sensitive or "active." Some wells in tight ground appear very sluggish. However, many apparently "dry," inactive, or "lost" wells can be rendered active by auguring below the casing, punching out the plugged perforations with a nail in a stick, or "shooting" the bottom of the pipes with a quarter stick of dynamite. Many holes considered "dead" and hopeless frequently became "active" with such treatment. Unremitting care and perseverance are necessary to obtain complete and dependable readings and effective records. Seemingly almost everything possible conspires to prevent a complete, full record of water levels being obtained, and it is well to approach such an investigation with that idea and attitude; otherwise disappointments and poor records almost invariably result.

As the casing of well SB-56 was cut off at slightly lower elevation, to discharge properly into the weir box, well SB-22 dropped to El. 570.81, the lowest yet recorded for this well, thus permanantly insuring that there will be no upward pressure under the toe of the rolled earth fill.

The flow of well SB-56, as of November 25, 1940, had decreased to 85 gal per min with the reservoir only 1.1 ft below normal level of 594, which further seems to indicate that at least a substantial part of the total flow does not come from the reservoir since greater fluctuations than this have occurred without affecting the water levels in the wells appreciably. By long-time observation of the flow from well SB-56, if effect of heavy rains, barometric variations, etc., can be properly compensated for, some interrelation between reservoir level and the wells may of course yet be proved. However, there seems to be much greater weight of evidence that there is little connection.

The system of wells drilled into the rock seams, particularly the key well SB-56, obviously situated so as to tap the "main stem" of the pressure system of seams below the dam, with weir box and recorder, is believed to constitute, for an earth-fill dam over seamy and cavernous limestone, an ideal arrangement of drain holes or pressure relief holes, equivalent to that generally considered ideal for a concrete dam.

It is possible that a part, or the whole, of this study and investigation, particularly now that the satisfactory outcome is known, could have been

considerably curtailed, but the writer is strongly of the opinion that, where an investment of $25,000,000 to $30,000,000 is involved, the comparatively small cost of such an investigation is definitely and fully justified, and to omit it, in view of past and recent history of dams and other engineering projects, should be considered as not good engineering practice.

In his discussion Mr. Moneymaker notes the disproving of the concept stated to have been long held by geologists to the effect that cavities may not be formed below the water table. The writer has been curious as to the line of reasoning, or experience, which originally gave rise to this concept. From an engineering viewpoint it would seem evident that ground water, in constantly seeking its level, lower outlets, etc., would frequently circulate through joints, bedding planes, or other orifices or pervious strata far below any existing or former free water table. The writer would think that such a theory would have long since been disproved by such known facts as the leakage channels under Hales Bar Dam below the original stream bed, the "lost" rivers of many parts of the United States, the many rising springs which apparently come from channels below the water table, etc. In the writer's opinion, drill holes, adequately deep and numerous, are the main evidence upon which final dependence can be placed in structures of major importance.

Quite properly, Mr. Moneymaker stresses the great advantage of early recognition of all conditions that threaten subsequent leakage, and the "cost and difficulty of stopping, or even appreciably retarding, leakage that has developed in limestones or dolomites under and around a large dam." The writer has frequently noted what is thought to be considerably too much complacency among engineers in regard to the possibility of fixing leaks after they once start. Most sites are capable of fairly certain diagnosis upon a thorough study of an ample number of carefully located drill holes and a study of the behavior of a reasonable number of ground-water observation wells, etc., plus perhaps a certain amount of practical engineering and geological intuition. Even in loosely, partly filled, and widely ramifying limestone caverns and seams, an appropriate amount of cost and time in previous investigation, grouting, and blocking will remove all reasonable possibility of serious leakage, and similar methods will diagnose and prevent (as described in this Symposium) serious leakage through pervious overburden. However, to further Mr. Moneymaker's thought, after several thousands or hundreds of thousands of acre-feet of stored water have torn through these same channels, due to having been left partly or totally uncorrected, it is certain to be a much different and much more serious matter.

Mr. Moneymaker appropriately mentions the early suspicion with which the Guntersville (Bangor limestone) formation was regarded and the considerable leakage under the only previous dam founded upon this formation (presumably Hales Bar Dam already mentioned by the writer). He also mentions that the largest caves in the entire TVA area occur in this formation. Mr. Ross, and certain associates, explored some twenty-five or thirty such caves in the Guntersville reservoir walls. One of these, Honeycomb Cave, completely by-passed the right abutment below pool level. Except for fortuitous circumstances and conformation, this large branching cave might have defied blocking-off opera-

tions, even at excessive cost. Great subsequent cost was incurred at Hales Bar, with only partial success, because of caverns and seams which were probably not so large. Another cave of almost comparable size, buried under the south cutoff, was discovered only after construction was well advanced and after many staff consultations had been held as to the advisability of not doing anything further than to rely on the steel sheet piling already driven.

The foundation problems, which Mr. Moneymaker (partly quoting Mr. Ross) describes as being serious only where there were solution cavities, actually involved, in addition to a complete steel sheet pile cutoff through the pervious gravel, about 2,100 lin ft, in the rock, of the approximately 4,000-ft length of the dam axis. Whereas the remedial treatment in the last analysis might be characterized as mainly excavating to good rock and grouting, it actually involved a number of extensive and highly expensive construction operations to remove masses of boulders and get down into the rock and correct the seams and caverns. All these were possible only with heavy pumping and progressive backing out, while progressively backfilling with what Mr. Moneymaker apparently refers to as "impervious material." In all cases this material was solely a good grade of concrete and grout. This work was anything but simple and inexpensive as might possibly be inferred from the brief reference by the discusser. It is possible that Mr. Moneymaker writes only in the geological sense, without reference to the extremely costly preparatory and coincidental construction and permanent corrective work.

It is noted that Mr. Walter[35] found that a fine sand could be added to neat cement grout (evidently without admixture) for injection into solution cavities by means of the usual piston-type pumps. There has been some experience described where sand-cement grout has produced excessive wear on the pumps, but this effect apparently was not sufficient to cause Mr. Walter to comment thereon. Also, it is noted that Mr. Walter apparently leaned rather decidedly toward pumping grout into a hole as long as it conceivably could do any good, rather than to worry too much about possibly wasted grout, since a closely parallel hole frequently failed to connect with the usually irregular open area under treatment. The writer is in full accord with this general policy and has occasionally wished some "prolific" hole had not been allowed to "freeze up" slowly, especially after subsequent holes failed to find the irregular system of open seams. Mr. Walter's experience also proves the tight results obtained by regrouting, where necessary.

The writer agrees fully with Mr. Prouty's belief that projects involving great expenditures and responsibilities should have every possible check as to permanence and safety. Mr. Prouty's suggestion as to deep artesian effect is also very interesting.

Mr. Feld very properly suggests that it would aid the reader if the total cost of exploratory work and grouting for each of the dam projects was given. If anything beyond generalizations are to be attempted, it would be necessary to make a detailed study of each job in order to compare efficiency, results, etc., of different methods, as cost keeping and other procedures vary more or less in spite of standardization.

---

[35] Mr. Walter died on June 30, 1940.

A rather complete ground-water study, such as was instituted almost from the beginning at Guntersville Dam, will, in large measure, answer Mr. Feld's question as to what changes downstream of the dams are to be expected from complete stoppage of the subterranean flows. However, since permanent tailwater will practically coincide with original river levels, conditions below the dams should change very little.

No admixtures of any kind were used with cement grout at Guntersville Dam at any point where watertightness or permanence was sought.[35a] The writer has grave doubts as to the wisdom of using any known and economically or practically feasible admixture in connection with any permanent part of structures of such importance and magnitude. A certain amount of fine sandy

FIG. 69.—CEMENT GROUT THAT HAS IMPREGNATED SAND, GRAVEL, AND MUD (HYDRAULIC CELL FILLING AND RIVER-BED MATERIALS)

silt-cement mixture was used at Guntersville Dam in an attempt to fill certain rock cavities upstream and downstream of the cutoff (the latter already made tight and permanent by various methods described, all using a good grade of concrete or neat cement grout) of the south earth embankment to obviate possible cave-ins. The cement in this case was more or less of a gesture toward slightly more insurance, if possible, that the material, which was the natural material of the flood plains, would "stay put." When, similarly to the experience reported by Mr. Walter, the difficulty of finding the ramifying seams,

[35a] Correction for *Transactions: Proceedings*, June, 1940, p. 1206, line 15, change "'Ross'" to "'Lewis'."

with holes placed where the seams were estimated to be, became apparent, and a question of watertightness also became involved, the admixture of sandy silt was discontinued and only pure cement grout was used thereafter.

The writer agrees with Mr. Minear on much of his discussion, particularly that a larger number of the smaller diameter, and usually cheaper, grout holes generally will result in the most efficient and economical over-all results. It is found, as also reported by Mr. Walter, that the bulk of the grout usually is taken by a comparatively few holes which happen to pierce the seams or achieve advantageous openings in them. Also, frequently, due to interconnection, several holes, which often may be large-diameter expensive ones, may be wasted if drilled previously.

Mr. Minear's experience with grout coming in contact with seam filling material is noted. The writer has exposed certain areas where pure cement grout came in contact with dredge filling of gravel, sand, and mud as in Figs. 39 and 69. Subsequent opportunity for close examination revealed that the cement had everywhere retained its integrity completely, shoving the mud into pockets and interstices, and occluding it, so that the mass was hard and watertight, resembling concrete, with frequent isolated pockets of foreign matter. It also followed and compacted many white seams in residual clay, forming series of irregular watertight cement cells.

After all, it is a matter of engineering judgment as to method, probable results, and how much money should be spent in the initial operations of drilling and grouting. Previously cited papers on this subject, published in 1934, 1935, and 1936, are believed to be illustrative of some of the points discussed.[34]

Acknowledgments.—Grateful acknowledgment is made here to Henry D. Harris for unremitting effort, interest, and care in gathering data and for assistance in preparing the part of this paper dealing with the ground-water investigation, and to Mr. Ross for a careful study of all geological features and assisting the writer to a better understanding of them.

PORTLAND P. FOX,[36] JUN. AM. SOC. C. E. (by letter).[36a]—Mr. Marsh's warning to others who may attempt to build a dam on a soluble limestone foundation is well fitted into these discussions. Not all dam sites on limestone are necessarily bad, but any dam site on a limestone foundation should be regarded with much suspicion until it is proved to be satisfactory.

Perhaps too much stress cannot be put on the need for closely spaced and deep exploratory core-drill holes in a limestone foundation for a proposed dam. At least ten times as many core-drill holes may be required to prove a dam site on a limestone foundation as on most other types of rock foundations. In the TVA region a number of possible dam sites on limestone had been explored, previous to the creation of the Authority, by a few widely spaced and shallow core-drill holes, and most of the sites appeared to be nearly perfect from these few holes; but on closer drilling and with deeper holes many of them proved to be very unsatisfactory. It is far better to explore a dam site on limestone thoroughly than to do a halfway job and encounter many costly surprizes after construction has started.

---

[36] Geologist, TVA, Spring City, Tenn.

[36a] Received by the Secretary January 6, 1941.

Mr. Marsh's discussion in regard to a dam founded on a soluble limestone as a doubtful safe long-time proposition is supported by a number of cases in which dams on limestone have developed serious leakage a few years after completion. In all cases it appears that the cause of the leakage has been due to the washing out of clay, sand, and other soft material that partly filled the pre-existing cavities or solution channels in the limestone and not to any rapid solution of the limestone itself. It should be remembered in regard to rates of solution that concrete is more soluble than most limestones.

If all of the cavities are thoroughly cleaned and completely plugged with concrete or grout, as they were at Chickamauga Dam, no leakage can possibly occur. In order for a limestone to dissolve, water must come in contact with the rock; and there must be circulation of the water or it will become inactive. If all of the joints, seams, and other openings in the limestone under the dam have been sealed at some place so there can be no circulation under the dam, there can be no solution of the limestone. The entire success of any foundation on limestone under a dam may depend upon blocking all of the openings completely. If this is not done, as Mr. Marsh has pointed out, leakage (even unnoticeable at first) may develop over a period of years and may become serious.

At Chickamauga Dam a great number of drill holes have been located strategically downstream from the earth dams, and at several saddles in the reservoir, for ground-water observations. The ground-water conditions before and after filling the reservoir were observed and will be continued indefinitely by the Hydraulic Data Division of the Authority. Any changes in the ground-water conditions should be detected quickly.

JAMES B. HAYS,[37] M. AM. SOC. C. E. (by letter).[37a]—Mr. Simonds requests information as to the method used in grouting with asphalt or pitch. A 6-in. casing was seated in the rock and extended just above the surface of the ground, and 5½-in. holes were drilled into the cavity to be grouted. A 2-in. steam pipe, capped on the bottom, extended to within a short distance of the bottom of the cavity; and inside this was a slightly shorter 1-in. return pipe. The lower ends of the steam pipes had to be kept free to allow for expansion. The casing head was piped to receive melted asphalt between the 2-in. steam pipe and the outside casing. Steam was piped into the 2-in. pipe, which was brought up through the casing head, and was discharged from the 1-in. return which projected above the 2-in. pipe. Asphalt was heated in portable kettles of the type used by roofing contractors. These kettles have a built-in rotary pump which, in this case, did not stand up under long-continued use. Small, piston-type, air-driven pumps were made locally and used to force the asphalt into the cavity. These pumps had a cylinder 1 in. in diameter and were arranged so that the piston stroke could be varied from ½ in. to 6 in. The pumping rate varied from 1 to 2 bbl per hr. When asphalt is used the kettles soon become caked with carbon. Pitch does not cake.

In operation during cold weather the steam is turned into the heating pipe in the hole for a short period to warm it up before pumping of the asphalt or

[37] Constr. Engr., Kentucky Dam, TVA, Gilbertsville, Ky.

[37a] Received by the Secretary December 9, 1940.

pitch is begun. Since the cavities to be grouted had water in them, a vent pipe was also provided at the casing header to release any steam pressure created during the introduction of hot asphalt. After the grouting is well started, the steam heat can be discontinued. During warm weather it was not necessary to use steam heat. If grouting has been stopped and it is later decided to resume operations, the steam is used to soften the material in the casing, after which grouting can continue.

Rapid pumping causes the asphalt or pitch to spread out in a thin layer or stringers. Slow pumping causes the formation of a large bulb, the outside shell of which is tough, being formed by cooling action in the water. The center remains hot and liquid, and there is considerable shrinkage when it finally cools off. Subsequent regrouting will eventually reduce the space left by shrinkage. For thin seams the rapid pumping gives greater coverage. To stop flows through large openings, the slow bulb-forming process seems to be more effective.

Mr. Simonds questions the stability of bentonite grout mixtures. At Chickamauga Dam it was used only for temporary construction grouting or for backfilling open cavities under the earth fill. Construction grouting consisted of blocking off flows of water into areas to be excavated later. This operation was highly successful. No dependence was placed on it for a permanent water seal. Small percentages of bentonite have been added to cement for permanent grouting on other projects, such as the Agency Valley Dam in Oregon, built by the Bureau of Reclamation. There is an increase of volume due to the addition of bentonite to cement, and a smoother mix results. It will crack or shrink on drying in warm air, but underground no doubt it will remain in good condition.

The clay-cement grout was used for temporary or backfill purposes only at Chickamauga Dam. However, it must be remembered that only certain types of clay will make a satisfactory grout. Topsoil is often used with cement to grout under concrete highways where settlement has occurred to bring the surface back to its former elevation.

Mr. Minear has emphasized several important points. The writer agrees with him that the rule of grouting until a hole refuses 1 cu ft of grout in a certain specified time is indefinite, inasmuch as it does not indicate the quantity of cement involved. The "second criterion" mentioned by him of pumping "to refusal at a pressure which is a fractional part (such as two thirds) of the allowable pumping pressure" might be misunderstood. Assuming that a hole is to be grouted to 100-lb pressure, his statement would mean to stop grouting with refusal at 66 lb. In conversation with Mr. Minear some years ago, the writer understood that it was his idea that pumping should be done at a higher pressure than that specified, in which case the refusal at a lower or fractional pressure would give the desired results. This conversation occurred at Hoover Dam during the construction period, and high-pressure methods were under discussion. The writer agreed with him. Other foundation conditions where pumping could not be done at higher pressures than the desired refusal, due to uplift, would require a modification of the rule. In either case the final water-cement ratio at refusal should be specified.

In regard to handling grout at a slow rate, the consequent settling out in the supply lines can be largely overcome by the use of a circulating system, as mentioned by the writer under "Grouting Equipment." Of course this does not prevent settlement of cement in the grout in the hole itself; but it does bring well-mixed fresh grout much closer to the point of use than a single direct pipe line. Where a large enough hole is used, a header can be arranged with the grout pipe feeder extending to the bottom of the hole through the larger casing. Grout is fed through this pipe and flows upward and on through a connection to the casing back to the pump sump. This method permits washing out the entire hole where and when desirable or necessary. This system was used at Chickamauga Dam and was mentioned by James S. Lewis, Jr.,[38] Assoc. M. Am. Soc. C. E. Costs are not yet available on the Chickamauga Dam, pending the completion of the final report on the project.

At Chickamauga Dam, in the area downstream from both embankments, wells were installed for observing the water table. Filling of the reservoir was started about January 16, 1940, and normal pool elevation was reached about April 1, 1940. To date (November 1, 1940) no fluctuation of the ground water has been caused by the reservoir.

[38] "Large Core Drills Aid Construction at Chickamauga Dam," by James S. Lewis, Jr., *Transactions* Am. Soc. C. E., Vol. 105 (1940), Fig. 6, p. 857.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE GRAND CENTRAL TERMINAL IN PERSPECTIVE

Discussion

BY MESSRS. ARTHUR V. SHERIDAN, AND C. E. SMITH

ARTHUR V. SHERIDAN,[28] ASSOC. M. AM. SOC. C. E. (by letter).[28a]—The evolution of the Grand Central Terminal, for a generation the world's most famous and interesting rail terminus, is a fascinating chapter in the career of some notable men. The talents, efforts, disappointments, perseverances, and travail of many forceful and imaginative minds are as truly a part of the great terminus and its associated projects as are the materials embraced in the monumental center which today invokes the admiration of those who view and the commendation of those who analyze it.

The brief yet adequate manner in which the story of the railroads, whose expansion lies behind the development of the Grand Central Terminal, has been told leaves the commentator little opportunity for any remarks other than an appraisal of a scholarly recording: By his concise presentation of salient facts, his recital of leading personages, their dreams and their rôles, and the exposition of the problems, trials, criticisms, and difficulties encountered en route, the author has produced a treatise interesting and instructive to any one who reads it. Although it is probable that most of the material contained in this paper is available in record form, the task of gathering and compiling it is undoubtedly one of considerable proportions. The manner in which this has been accomplished, including as it does personal records and recollections, gives documentary value to the paper.

It is particularly as a planning achievement that the Grand Central Terminal and its related features merit review. Individual parts of this project have their counterparts in many locations, in some instances even in more advanced forms; but it is doubtful if there presently exists anywhere a composite project more completely and adequately conceived and executed. No plan

NOTE.—This paper by William J. Wilgus, Hon. M. Am. Soc. C. E., was published in October, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1940, by Messrs. F. Lavis, E. R. Hill, Alonzo J. Hammond, and H. L. Ripley; and January, 1941, by Messrs. A. J. Meehan, and J. P. Hallihan.

[28] Planning Commr., City of New York, New York, N. Y.

[28a] Received by the Secretary December 28, 1940.

can be evaluated until it has been consummated. No person deserves the appellation of planner until his plans have been put to the test. In the planning of the Grand Central Terminal, almost every social, economic, and physical factor entering into daily urban life had to be weighed, located and judiciously incorporated in a form intelligible to those entrusted with the engineering, architectural, financial, legal, and administrative features of a travel, residential and business community, all located within one great indoors.

The writer recalls much of what has taken place during the past 45 years. As a boy in old Kings Bridge, N. Y., he daily watched the trains that sped over the rails passing between Spuyten Duyvil and the Grand Central Station and as a youth he used them en route to seek an education. He still can visualize the cut through solid rock that served as a railroad path for what has since been incorporated into the present pattern of Kingsbridge Avenue, Broadway, and Putnam Avenue, adjacent to West 230th Street and West 231st Street in the Bronx. The rerouting of the railroad, a well remembered undertaking, along the north bank of the Harlem Ship Canal, eliminated dangerous crossings and reduced to a mere memory a once active suburban railroad center jointly served by the New York Central and the New York and Putnam railroads and long since replaced by nearby intensive developments.

It is interesting to note how expansion of the City of New York has paralleled that of the railroad. The early movement northward of the terminus of New York's first railroad followed the trend of population on Manhattan Island. By the time the present Grand Central was in the process of final planning, New York had been greatly augmented by the formation of the Greater City, which brought Kings, Queens, and Richmond counties into the perspective of future plans. The center of population was no longer to move north but rather to the east. In view of its geographical position, 42d Street was well envisioned as a permanent terminus for a railroad connecting New York and the great centers of the east, north, and west. In this, as in other respects of its planning, the project has demonstrated the foresight of its authors.

There can be little question that the local electrification of the railroad was one of the great milestones of its existence. To this accomplishment, both public and corporation can attribute a relationship that has permitted mutual expansion. It is inconceivable that a Grand Central Terminus could have remained where it is, or in fact within the city at all, had former methods of propelling trains been retained. That science and engineering have made possible a speed, cleanliness, smoothness, and safety of travel undreamed of, even a half century ago, is obvious to all who have witnessed the progress of intervening change; nor is the end in sight.

The effect of time, from the viewpoint of finances, upon the execution of a project, is vividly portrayed in the accomplishment presently being reviewed. Great as was the cost of the project planned about 1900 and substantially completed within the decade following, the sum was only a fraction of what would be required to execute it at this writing.

In comparing the original plans with present conditions, one is constrained to the opinion that it is somewhat regrettable that the proposed Court of

Honor at the north end of the main station building was eliminated from the project. The esthetic value of the great structure facing Park Avenue would have been considerably enhanced and the approach to the elevated roadways encircling the building would have been less suggestive of a tunnel hazard to an automobile en route.

Of particular interest to the writer and to a community of 1,500,000, known as the Bronx, is the suggestion for a great civic and business center in the vicinity of 149th Street and the New York Central right of way. From the standpoint of public service, it is indefensible that the Bronx should be without facilities for receiving and discharging passengers from cities served by the two railroads, none of which centers, except Chicago, Ill., has a population equal to that of the Bronx and its neighbor Westchester. Included in the original plan of Colonel Wilgus, this project was advanced to a stage which convinced both public and elected officials that certain associated city planning was necessary. Two subway routes now reach the location, a central post office has been erected, Mott Avenue has been widened and renamed the Grand Concourse, and a magnificent civic and county center, with a modern hotel, located within a half mile has been built.

Several years ago, James J. Lyons, President of the Borough of the Bronx, made overtures to the railroad in the hope of reviving the project. When these failed, the City of New York instituted legal proceedings to compel action. That the railroad should be reluctant to proceed with its plan of nearly forty years standing will probably be attributed to financial difficulties. However, when one reflects that the population, which it would serve, has increased from less than half a million to more than two millions, since it was first deemed desirable, it is difficult, even from an economic standpoint, to justify a postponement of so many years. The inconsistency of the railroad policy is particularly confusing to those who, like the writer, recall the directional sign and passageway installed at the instigation of the railroad for the purpose of leading passengers to and from the subway at the Mott Haven terminal station. Barring perhaps the intense concentration of people and business, practically everything that has been possible at Grand Central could be duplicated at Mott Haven.

A proposal for covering the Harlem Branch of the railroad north to Fordham, including the reclaiming of adjacent blocks, has been developed, and a Park Avenue comparable in distance and superior in width and treatment to that north of Grand Central awaits only financial and legal impetus. Preliminary plans and estimates for connecting 96th Street and Park Avenue in Manhattan with the Grand Concourse and with Park Avenue, by means of an express highway, have also been made with the approval of the Board of Estimate. Both of these projects were developed under the direction of the writer as chief engineer of the Bronx and for five years have been part of a borough arterial highway program. Although it is generally conceded that the most desirable treatment for Park Avenue south of the Bronx would be to depress the railroad, it is also recognized that placing the New York Central tracks north of 96th Street beneath Park Avenue would require tunneling in Man-

hattan and under the Harlem River as well as relocating, realining, and regrading both the New York Central and the New Haven tracks for a considerable distance in the Bronx at a cost variously estimated at from $200,000,000 to $350,000,000. The present proposal contemplates maintenance of the existing track system, which would be flanked in Manhattan by through and local express highways, with service roads. These highways would be treated and landscaped in such a manner as to eliminate any objectionable features to the adjacent proposed residential area, which will be sufficiently distant and screened from the railroad to provide, for most of the distance north of 96th Street, a prospect ever more inviting than that to the south. Including the acquisition of property on Park Avenue, making an artery 340 ft wide, an ornamental bridge across the Harlem River and a treated viaduct in the Bronx, the total estimated cost approximates $40,000,000, a major part of which would be recouped from increased land and building taxes. The further acquisition of an entire block on each side of Park Avenue, for park and residential purposes, has also been considered. The cost of such additional land would increase greatly the cost of the project, which is presently contemplated to be a self-liquidating toll highway.

It may be that the consummation of this proposal, as well as that of a Bronx Railroad terminal, will have to await the evolution of a new economic system wherein a nation possessed of a recognized need for improvements, a plentiful supply of man and machine power (much of which is still unemployed), an almost inexhaustible supply of material, an increasing fund of knowledge, and an engineering talent and ambition equal to that of any previous generation, will not be compelled to forego urgent projects for the betterment of mankind because of a stated deficiency of a medium of exchange presently termed money. The writer contemplates both a terminal and a highway as well as a new economic order in the not too distant future.

In one instance, an expressed conviction of the author may be open to question. The tremendous increase in land values in Westchester County, in the opinion of certain groups, is due rather to its parkway system than to railroad facilities. Almost identical claims are made by the proponents of rail and highway influences. Whatever the principal cause, there is ample room for the important rôle played by each. In fact, the cause of growth might be given a time index; the earlier expansion belonging almost exclusively to the railroad impetus and the more recent increase to the Westchester Parkway facilities.

It is eminently fitting that the paper should quote the distinguished engineer, then chief advisor to the Board of Estimate and Apportionment, who, in recommending that the city approve the plans submitted by the railroad, envisioned in his mind's eye the world's finest railway terminal. This vision long since has been realized. As an apostle and an authority on comprehensive municipal planning, the commendation of Nelson P. Lewis was, in this instance, a compliment as much to be valued as merited, and the tribute paid is but a suggestion of how an engineer can aid in giving due public acknowledgment to the talents and achievements of a fellow member of his profession.

It is not in the accomplishment of the Grand Central development alone that the vision and talent evidenced by this paper are apparent. The author's proposals for transportation facilities adequate for the needs and possibilities of the Metropolitan area, and his proposals for a New Jersey-Long Island Railroad freight tunnel and a connection between New Jersey and a point north of Manhattan, as well as certain other suggestions for improving Metropolitan services, land, water, and air, merit the consideration of planning agencies and public officials. It is from the dreams of imaginative men that mankind progresses. Colonel Wilgus is one who has witnessed in a goodly measure the realization of some of his dreams.

C. E. SMITH,[29] M. AM. SOC. C. E. (by letter).[29a]—Civil engineering is shown at its best in the conception, planning, and consummation of such a great engineering work as the Grand Central Terminal, described by Colonel Wilgus.

The profession is fortunate indeed that Colonel Wilgus has given so much of his time and thought to gathering together, in such interesting fashion, what might otherwise have been rather dry statistics relating to one of the most important engineering projects of modern times. The paper is an important contribution to the history of the New York and Harlem, New York Central, and New York, New Haven and Hartford railroads, to the history of the City of New York, and to the history of civil engineering.

One important feature that was not mentioned in the paper relates to the ramp system which distinguishes the Grand Central Terminal from all other passenger terminals in the United States. Several years ago the writer had occasion to look up the background of that system and traced its inception back to Colonel Wilgus himself, who, in the absence of other evidence, is entitled to the entire credit for the inception and conception of the ramps which have relieved so many millions of passengers of the onerous task of climbing and descending stairs. That one feature alone is worthy of the greatest commendation.

There is one feature that has been developed since the construction of the Grand Central Terminal that would have added very materially to the convenience of passengers, particularly commuters, using that station, and that is the mezzanine between street level and track platform level. This has been developed quite successfully in the underground commuter terminal of the Pennsylvania Railroad at Broad Street, Philadelphia, Pa., and more recently in connection with the Sixth Avenue Subway of the City of New York. Such a mezzanine at Grand Central Terminal would have eliminated the single unsatisfactory feature that forces some passengers to walk twice the entire length of trains between the waiting room and the farthest cars to and from 45th and other streets. A mezzanine at Grand Central would have permitted passengers to enter and leave train platforms as far north as 45th Street without the necessity of passing the gates at about 43d Street. The convenience to commuters in the 45th Street district would have been great, but it is too late

[29] Vice-Pres., N. Y. N. H. & H. R. R., New Haven, Conn.

[29a] Received by the Secretary December 23, 1940.

now to install the mezzanine except at an expense out of all proportion to the benefits that would follow.

Referring to the different systems of electric traction adopted by the New York Central and New Haven railroads, Colonel Wilgus states (see heading "Grand Central Terminal Transformation—Formative Period from 1899 to 1907: Concept of All-Electric Service Throughout Suburban Region"):

> "The A.C.–D.C. struggle later was to assume the dignity of a 'celebrated case' comparable to the famous 'battle of the gages' in England in bygone years. It gathered force when the New York, New Haven and Hartford Railroad, owning trackage rights south of Woodlawn, belatedly adopted the views of Mr. Westinghouse and found it necessary to plan an electric locomotive that could operate on both alternating and direct currents."

The electrical engineers and management of the New Haven Railroad Company were bold indeed to venture into a new and relatively untried system of electrification—11,000-volt, 25-cycle, single-phase alternating current, which is carried directly to the locomotives and multiple-unit cars through the pantographs—particularly as it is also necessary to operate its trains on 600-volt direct current on the New York Central tracks between Woodlawn and Grand Central. However, continual research of the New Haven Railroad Company, in cooperation with the Westinghouse Electric and Manufacturing Company, has developed that system as the preferable system for long-distance, heavy-traction, railroad electrification. The New Haven-Westinghouse system has been adopted almost universally—with notable exceptions—throughout the world for heavy, long-distance electrification of railroads. The most recent convert to that system is the Pennsylvania Railroad Company in its electrification between New York, Washington, D. C., and Harrisburg, Pa., following the Great Northern, Norfolk and Western, Reading, and Virginian railway companies. If the New York Central Railroad Company were adopting a system of electrification today, it is doubtful if it would adopt its present system of third-rail, 600-volt direct current.

Colonel Wilgus' reference to a proposed new station at 149th Street and a loop just south of it is interesting indeed, particularly as interest has since been directed to that possibility by construction, underground at that point, of a two-level, rapid-transit station where express trains of the 7th Avenue and 4th Avenue subways are available; but the load on Grand Central has not yet reached the point where an auxiliary intermediate terminal at 149th Street would be justified.

Colonel Wilgus refers to the increase in assessed values in New York City and Westchester County, a considerable part of which, however, must be credited to the rapid-transit developments in New York City and to the highway development during the period. Just prior to the start of work on the present Grand Central Terminal, the city had placed in operation the first subway with loop terminal at City Hall, located on 4th Avenue south of Grand Central, on Broadway north of 42d Street, and running on 42d Street between 4th and 7th avenues. This was the first so-called "Z" subway, which was later enlarged into the east and west side subways by extending the 4th Avenue line

north on Lexington Avenue, with branches in the Bronx, and by extending the Broadway subway south on 7th Avenue. Both lines were extended to Brooklyn. The former connecting link between Grand Central and Times Square now constitutes "the Shuttle." There is no doubt that this expansion of rapid transit, taken in connection with the development of the Grand Central Station, as well as the development of express highways, had a profound effect on property values in New York City and in Westchester County, notwithstanding which the Grand Central Terminal must be given credit for a large part of the increase in property values, both in and around Grand Central and in Westchester County.

Indeed, it may be said that the electrification of the New York Central and New Haven railroads, and the building up of Westchester County, may have had considerable to do with the decline in population in the Borough of Manhattan south of 42d Street, most of which, however, must be attributed to the development of rapid transit.

Just preceding the heading "Beneficial Results: Public Blessings," Colonel Wilgus shows a tabular comparison between passengers in and out of the Grand Central Terminal in 1930 compared with 1906. Under "Résumé and Conclusion: The Future" he states: "It would seem, therefore, that the time is not far distant when, as was done forty years ago (1899), thought must be given to the taking of measures for handling a further increase of traffic and for satisfying public demands." Table 7 shows a comparison between passengers handled in and out of Grand Central Terminal in 1930 and 1939. In the latter year, the New York, Westchester and Boston Railway did not operate, having suspended all operations on December 31, 1937. Also shown for comparison is 1936, the last year in which the New York, Westchester and Boston Railway operated in full. In that year, the New York, Westchester and Boston Railway handled 4,220,138 passengers to and from the 133d Street terminal at the Harlem River.

Although there was a tremendous increase in the number of passengers handled from 1906 to 1930, there was a decrease of slightly more than 20%

TABLE 7.—Passengers In and Out of Grand Central Terminal (In Thousands)

Description	1930		1936		1939	
	Commuters	All	Commuters	All	Commuters	All
New York Central Railroad	26,264	33,765	15,824	21,437	16,481	22,470
New York, New Haven and Hartford Railroad	9,444	16,878	6,828	14,289	8,361	17,308
Total	....	50,643	....	35,726	....	39,778

from 50,643,000 in 1930 to 39,778,000 in 1939, even after including passengers using the Grand Central Terminal who formerly used the defunct New York, Westchester and Boston Railway. There are no indications at the present time of such an increase in traffic as would require any material enlargement of the

capacity of the Grand Central Terminal and, needless to say, the job was so well done that public demands are still quite well satisfied.

Although Colonel Wilgus attributes the increase of passengers over a long period, from 1906 to 1930, to electrification, it is interesting to note that other

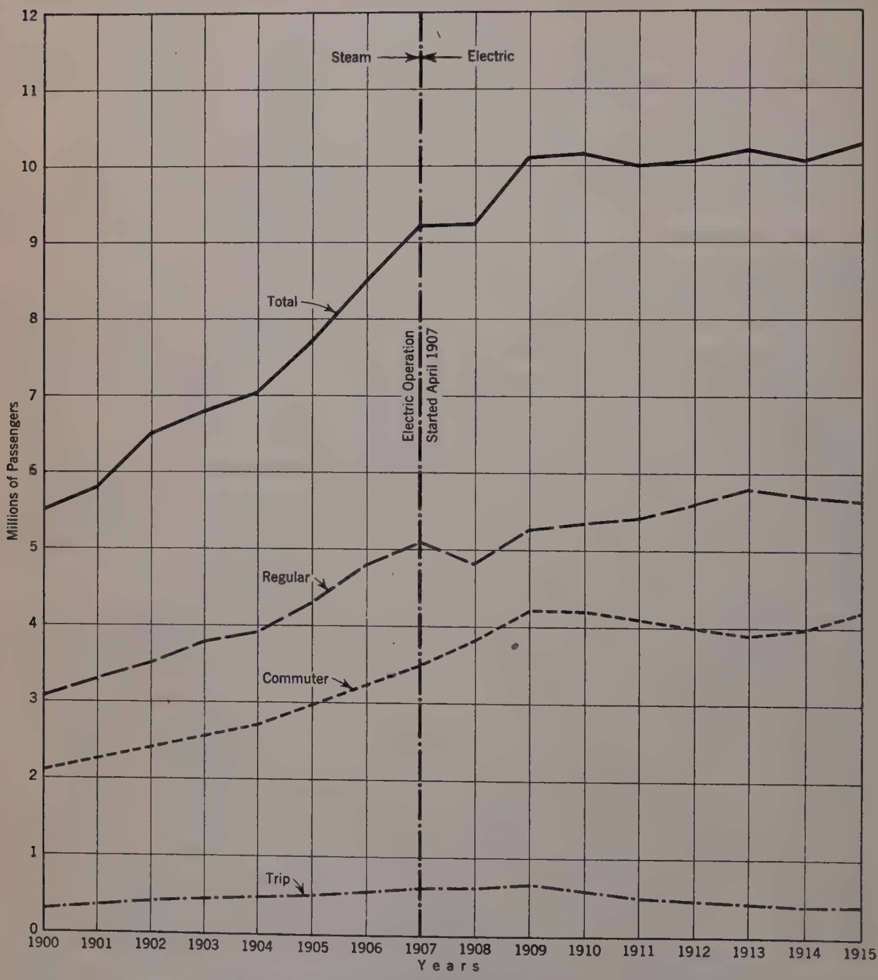


FIG. 15.—PASSENGERS ENTERING AND LEAVING NEW YORK CITY VIA THE NEW YORK, NEW HAVEN AND HARTFORD RAILROAD

factors have a profound effect on the trend of passengers carried. This is illustrated by Fig. 15, showing number of commuter and other passengers handled in and out of the Grand Central Terminal by the New York, New Haven and Hartford Railroad each year from 1900 to 1915. It will be noted that there was a rapid increase in the total number of passengers handled by steam locomotives from about 5,500,000 to slightly more than 9,000,000 in

1907, the year when steam operation was discontinued and electric operation started. This increased to approximately 10,000,000 in 1909. It remained at this level for seven years to and including 1915, indicating that electrification in and of itself was not sufficiently persuasive to continue the upward trend that had been in effect seven years prior to electrification. Of course, the depression of 1907 and the ensuing stagnation of business was evident in preventing an increase of passengers until 1915, following which there was a considerable further increase stimulated by the first World War.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

PRESSURE-MOMENTUM THEORY APPLIED TO THE BROAD-CRESTED WEIR

Discussion

By H. A. Doeringsfeld, Esq., and C. L. Barker, Assoc. M. Am. Soc. C. E.

H. A. Doeringsfeld,[36] Esq., and C. L. Barker,[37] Assoc. M. Am. Soc. C. E. (by letter).[37a]—In closing the discussion on their paper the writers wish to express their appreciation to the men who have so kindly discussed the results. Many valuable suggestions have been made and it is hoped that the information contained in the paper and the discussions will be of value in further study.

The observation made by Mr. Hackney as to the range of the ratio of $\frac{H}{B}$ is important. It emphasizes the importance of being able to obtain the position of "nearly parallel" flow. This observation agrees closely with the value given by Professor Prentice of $\frac{B}{H} = 5$ (or $\frac{H}{B} = 0.2$). He states that parallel flow existed only when $\frac{H}{B}$ was less than about 0.2.

The suggestion by Professor Prentice that some of the discrepancies in the values of Ω might be caused by the difficulties in obtaining the true minimum depths is undoubtedly correct. To read the water surface elevation to an accuracy greater than 0.001 ft is quite impossible; yet errors in d_3 of that size might affect the value of K appreciably, especially at the low heads.

Professor Bakhmeteff has emphasized quite well what the writers tried to state in cautioning about the use of Equation (18).

In their studies the writers considered the possibility of effect of the downward acceleration of the water as suggested by Professor Curtis. However, not knowing just how to account for the effect of this force in the computations

Note.—This paper by H. A. Doeringsfeld, Esq., and C. L. Barker, Assoc. M. Am. Soc. C. E., was published in December, 1939, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1940, by Messrs. J. C. Stevens, and H. G. Wilm; and April, 1940, by I. M. Nelidov, Assoc. M. Am. Soc. C. E., and June, 1940, by Messrs. John W. Hackney, Thomas H. Prentice, Boris A. Bakhmeteff, D. D. Curtis, Carl Rohwer, and John Hedberg.

[36] Asst. Prof., Mathematics and Mechanics, Univ. of Minn., Minneapolis, Minn.

[37] Asst. Prof., Hydr. Eng., Coll. of Eng., State Coll. of Washington, Pullman, Wash.

[37a] Received by the Secretary January 22, 1941.

it was neglected, as was also friction; and a study was made to determine what results could be obtained without considering these effects.

The fact that the true momentum cannot be computed by the use of the average velocity has been mentioned by several writers. Professor Curtis states how much in error the momentum, computed by average velocity, might be. A detailed study of the velocity distribution would be necessary to determine, accurately, the size of the errors that develop through the use of the average velocity.

Mr. Rohwer states that the broad-crested weir is subject to the same limitations as the sharp-crested weir in the field because of the settling basin for debris. However, if the crest height is quite large, the effect of small changes of crest height due to the deposit of debris is slight and quite satisfactory results can be obtained. Of course, a large layer of sediment behind the weir will affect the results.

On the Minnesota weir tests, the water-surface profile on the downstream end of the weir was taken and these data were carefully studied but not published. Professor Hedberg suggests that they might be of value. However, the surface was so rough that its mean plane was difficult to obtain. In the tests mentioned, the tailwater surface was raised from the lowest position possible to a position at which the "nearly parallel flow section" was forced upstream. Between these two positions the tailwater surface elevation had no effect on the upstream head; but when the "nearly parallel flow position" was forced upstream, the head above the weir was changed.

Professor Hedberg also mentions the interesting fact that the eddy overhanging the weir increases its effective height. This is an important observation and should be studied, as he suggests.

Messrs. Stevens, Curtis, and Hedberg mention the fact that if the slope of the crest is made other than zero, the equations will be changed. This is true, of course. The crest was made level in all tests to keep all equations as simple as possible. As they mention, however, if the crest had a slope the equations could easily be developed from theory. It is well that they stressed the point that the slope was zero so that some attempting to use the expression would not use it in error.

Mr. Wilm states that "Throughout the paper, apparently, the assumption is made that parallel flow will occur over the level weir surface, except where the flow is influenced by end conditions." This statement is scarcely correct. Certainly flow cannot exist in normal conditions with the bed of a channel horizontal and the water surface parallel to the bed. There must be a slope provided either to the surface, or the bed, or both. Perhaps the writers are in error in the use of the words "nearly parallel," for there is no degree of parallelism. In the "Conclusion," the statement was made by the writers: "The accuracy of the formulas is to be questioned if the breadth of the weir is so small that the water surface does not become parallel to the surface of the weir, or nearly so." This sentence meant to imply that, for a very short distance, if the stream lines could be considered nearly parallel to the weir surface then in that space, d_3 could be measured. In the case of raising the tailwater men-

tioned, as long as the position of measurement of d_3 was not shifted, there was no effect on the formula, and certainly, in the extreme high position of the tailwater, one could not consider flow, where the water surface at the d_3 position was concave upward, as a position of parallel flow.

A sharp-crested weir was used only because its very sharpness precludes the need for giving a radius of curvature. Certainly many weirs of different crest profiles could be studied, and perhaps better results obtained, since, by proper design, the vena contracta could be eliminated.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

DESIGN OF HINGES AND ARTICULATIONS IN REINFORCED CONCRETE STRUCTURES

Discussion

BY GEORGE C. ERNST, ASSOC. M. AM. SOC. C. E.

GEORGE C. ERNST,[4] ASSOC. M. AM. SOC. C. E. (by letter).[4a]—The comments of Mr. Eremin are appreciated and the writer will discuss the various points in the order in which they were presented.

No attempt was made to provide the complete development of the equations nor to state the usual structural assumptions, inasmuch as such material is obtainable elsewhere from readily available sources (2) (3) (11).[4b] Mr. Eremin may refer to such sources for the more comprehensive treatment of the analyses.

A careful study of the work of Messrs. Parsons and Stang (2), as well as that of the writer, will reveal that no claim has been made as to the exactness with which Eq. 1 represents the actual stresses within the hinge. The writer sees no objection to the development and use of a simpler formula, but it should be recognized that Eq. 1, by providing the maximum stresses, may be utilized to guard against the effect of repeated rotation when considered critical, as suggested by Admiral Moreell (3). In his comprehensive investigation of all types of hinges (22), Mr. Kluge took advantage of short hinge openings, with consequently small values of $\frac{L}{r}$ for the Mesnager hinge in order to assume that the full yield point of the steel might be developed. Tests by Mr. Kluge appear to show that repeated rotations are not critical in ordinary construction. The major objection to Eq. 1, as noted by the writer, is that it shows a non-existent advantage in the reduction of the maximum stress for the larger values of $\frac{L}{r}$, due to the increased flexibility of the bars. If a proper limit is placed on $\frac{L}{r}$ for various grades of steel, Eq. 1 will yield conservative values.

NOTE.—This paper by George C. Ernst, Assoc. M. Am. Soc. C. E., was published in April, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1940, by A. A. Eremin, Assoc. M. Am. Soc. C. E.

[4] Riverdale, Md.

[4a] Received by the Secretary January 21, 1941.

[4b] Numerals in parentheses, thus (2), refer to the corresponding numbers in the Bibliography, Appendix II, at the end of the paper.

The writer wishes to emphasize that, although Eq. 1 was developed for stresses within the elastic limit, nevertheless it may be compared satisfactorily with experimental data at the ultimate since the load at first yielding is not critically different from that at the ultimate for structural members subjected to compression and shear, provided that a proper limit is placed upon the slenderness ratio.

The comments on Fig. 3 with regard to the change in action of the Mesnager hinge due to the concrete covering are quite true. With a limit on $\frac{L}{r}$ consistent with the grade of steel used, an adjustment for the effect of the covering may be made as suggested by the writer without the need for expressing exactly the stress conditions. The added resistance to rotation caused by the covering was found to be negligible in the writer's work as well as in that of Messrs. Parsons, Stang, and Kluge. Fig. 4, however, does not pertain to the Mesnager hinge although Mr. Eremin's discussion implies that it does. The writer believes that the formulas presented for the Considère hinges of Fig. 4 approximate actual conditions more closely than does Eq. 1 for the Mesnager hinge.

In so far as the Considère hinge is concerned, it should be recognized that such hinges should never be used at stresses that would crack and peel off the cover over the spirals, although Mr. Eremin states that such a condition is generally true. Actual designs (9) (10) as well as tests (11) reveal that the working stresses and deformations on such sections are well below those at which cracking and peeling of the cover exist even at the outer fibers of the high compression side when subjected to normal rotations. It seems a bit fantastic to talk of the increased strength of the core of a spiral and then choose a working stress which prevents the utilization of such properties. As a matter of fact, the plastic yielding of the Considère hinge, if subjected to such extreme conditions over the entire section, would prohibit the use of the units (11). The recommendation of the writer to keep within an allowable limiting deformation at the outer fiber of the high compression side when subjected to maximum rotation is conservative and permits the greatest use of the deformation ability of the concrete that should be permitted. The proposed value of 0.002 in. per in. is a limit in complete accord with field tests (10). Such a limit for the outer fibers, however, will fall far short of cracking off the covering over the spiral.

Unless the hinge contains only one spiral, it will be simpler to cast the hinge in a rectangular section. This would normally be the case, inasmuch as three or more spirals are generally required in each hinge for an arch of the size requiring such construction. In any event, the spiral remains virtually inactive, except as shear reinforcement, for the working stresses and deformations used in the past, or for the limiting deformation recommended by the writer.

(22) **"An Investigation of Rigid Frame Bridges, Part III. Tests of Structural Hinges of Reinforced Concrete,"** by Ralph W. Kluge, *Bulletin No. 322*, Univ. of Illinois Eng. Experiment Station, March, 1940.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SUPERSTRUCTURE OF THEME BUILDING OF NEW YORK WORLD'S FAIR

Discussion

By Louis Balog, Esq.

Louis Balog,[5] Esq. (by letter).[5a]—The five Perisphere designs mentioned by the authors were submitted to the Construction Department of the Fair for consideration on April 12, 1937. One of these, designated as Design No. 1, was adopted for construction. The major part of the paper presents a detailed account of this design.

Design No. 1 consists of a steel framework and a separate outer covering. The spherical steel framework is composed of girt and meridian trusses of a depth of $\frac{1}{16}$ to $\frac{1}{8}$ of the outside radius. Assuming pin connections at the intersection of the meridians and girts the stresses in the framework, due to symmetrical loads, can be determined from equilibrium requirements. If the connections are rigid, the elastic deformation of the girts results in meridian moments. The magnitude of these moments is primarily a function of the moment of inertia of the meridians. The deeper the meridians the larger the moments will be due to elastic deformation of the girts. In case of n girts the meridian moments can be obtained by the solution of $n - 1$ elastic equations expressing the equality of the elastic displacements of girts and meridians at their intersections. The operations involved are not too formidable; but they still are not justified by the significance of these moments as compared to those that result from the curvature of the meridians between two intersection points. For unsymmetrical loadings, like those due to wind, the number of elastic equations increases so far beyond any reasonable limit that a rigorous investigation becomes impracticable.

To simplify the analysis the authors divide the sphere at the equator. They assume the upper hemisphere to be a statically determinate structure and obtain the stresses from equilibrium requirements. In the lower hemisphere the meridians were assumed to be cantilevers resting on a series of elastic supports, represented by the girts. These assumptions result in a sufficiently close approximation of the stresses in the framework due to axially

Note.—This paper by Shortridge Hardesty, M. Am. Soc. C. E., and Alfred Hedefine, Assoc. M. Am. Soc. C. E., was published in September, 1940, *Proceedings*.

[5] Engr. with Leon S. Moisseiff, Cons. Engr., New York, N. Y.

[5a] Received by the Secretary December 28, 1940.

symmetrical loadings, such as the dead and snow loads. The wind-pressure distribution, as given from tests, is not symmetrical to any plane containing the vertical axis of the sphere. It is not adaptable to mathematical expression that is suitable for the derivation of closed formulas for the wind stresses. The authors' analysis resulted in safe values for the stresses produced by the assumed wind-pressure distribution as comparative calculations made by the writer show. The diagonal bracing provided, and the large moment-carrying capacity of the deep meridian trusses, make this structure well fitted to carry unsymmetrical wind loads.

Although rigorous investigations indicated that the approximate methods used by the authors are suitable for design purposes, it is regrettable that no stress measurements were made on the structure as built. Such measurements would have given invaluable hints for correct simplified analysis, the only kind that is of genuine value in the design of this type of complex structure. This holds especially for the wind stresses. Extensive laboratory investigations were made to determine the magnitude and distribution of the wind loads. The stresses produced by these loads, however, were not ascertained by an experimental method. The authors' skilful estimate of the stresses in the framework due to wind would be of considerably greater value if measurements had shown how closely the actual conditions were approximated.

The problem in the design of the Perisphere was to create a structure which carries specified loads safely and has a smooth spherical surface. The latter requirement necessitates the use of a complicated outside covering on a steel framework. By the use of reinforced concrete, such covering is eliminated and the carrying structure can be formed readily to the desired shape and surface.

Design No. 5, a reinforced concrete shell structure, was proposed and designed by the writer. It differs, not only in its material but also in structural conception, from the other four designs, which were ribbed steel structures. The principal characteristics, design data, and quantities of this design are presented in the following:

The great carrying capacity of shells in general and that of shells of double curvature, like the sphere, in particular, would permit the use of very thin sections. In the upper two thirds of this spherical shell the thickness was determined by construction considerations rather than by the stresses, the safety factor against buckling, or durability requirements. The construction of a sufficiently precise formwork and the assurance of careful workmanship in placing the reinforcing steel and concrete constituted the difficulties that influenced the selection of the minimum thickness. These considerations resulted in the adoption of greater thickness for the upper part of the shell than that of existing permanent structures with similar radius of curvature.

Fig. 18 indicates the principal dimensions and stresses of the reinforced concrete sphere. The outside diameter is 180 ft. It is supported on eight round columns, 4 ft in diameter, arranged similarly to the steel design.

The thickness of the shell increases slowly from 3 in. $\left(\frac{1}{360}\right.$ of the outside radius$\left.\right)$ at the top where $\phi = 0°$, to 4 in. $\left(\frac{1}{270}\right.$ of the outside radius$\left.\right)$ at the

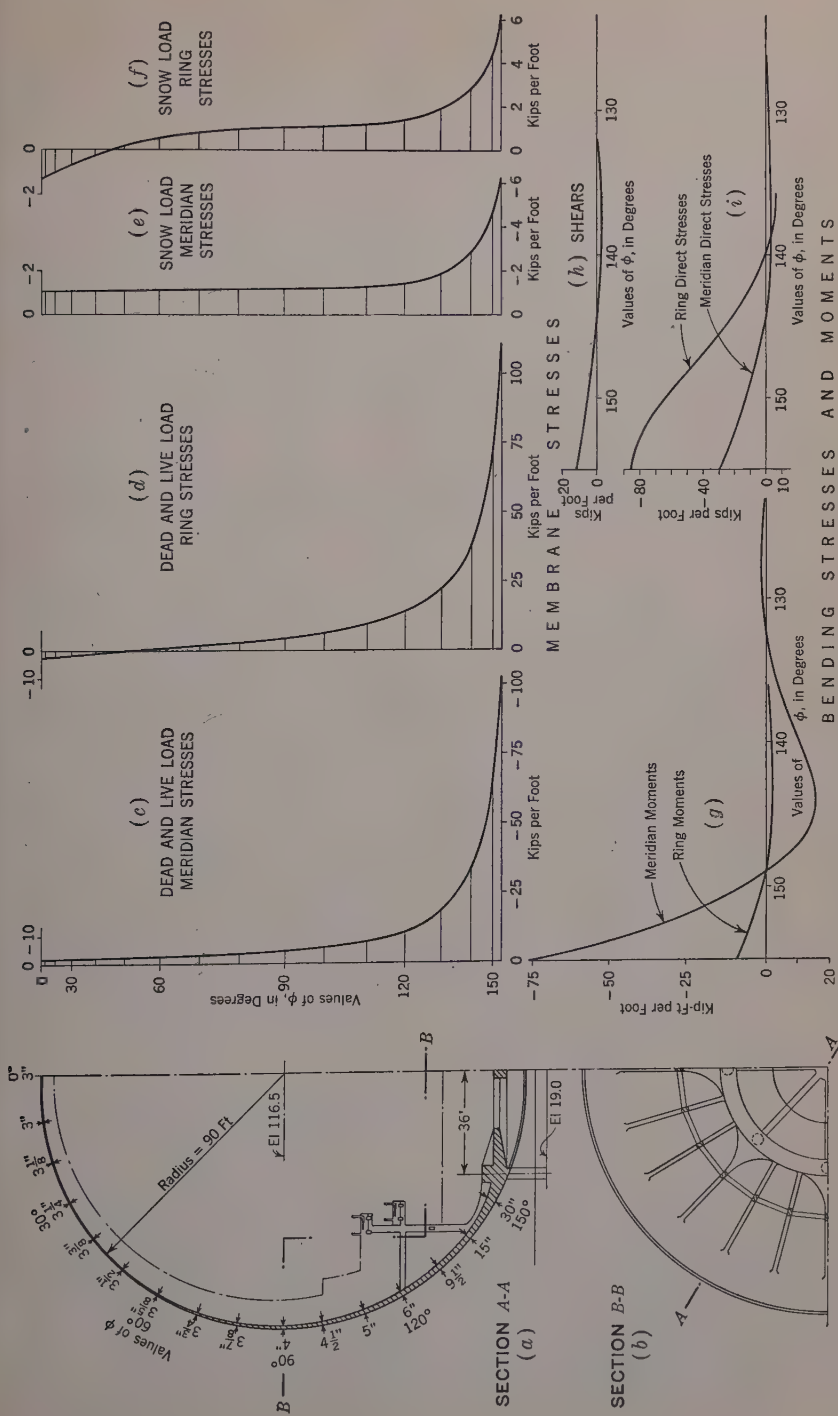


FIG. 18.—SECTIONS AND PRINCIPAL STRESSES OF CONCRETE PERISPHERE

equator where $\phi = 90°$. The increase in thickness below the equator is more rapid. At $\phi = 140°$ where the live load on the moving platforms, their weight, and that of their reinforced concrete supports are transmitted, the shell thickness is 15 in.—$\frac{1}{74}$ of the outside radius. Below this point the shell is stiffened by meridional ribs, its thickness rapidly increases and at the face of the support ring reaches 36 in.—that is, $\frac{1}{30}$ of the outside radius. The ratios of thickness to radius indicate that the membrane theory of shells, which assumes small thicknesses and uniform distribution of the strains in the cross section of the shell, is applicable to this structure except in the vicinity of the support ring. This assumption of uniform strains results in the disappearance of the bending and torsion moments perpendicular to the center surface of the shell and permits the calculation of the direct stresses, caused by axially symmetrical loadings, from equilibrium requirements.

The dead load and similarly distributed loads produce direct stresses in the direction of the meridians and parallel circles, causing small deformations of the shell. Rigorous investigation of spherical shells[6] indicate an upper limit of $\frac{\delta^2}{2\,a}$ for the magnitude of these deformations. For a shell thickness of $\delta = 4$ in. and a radius of $a = 90$ ft the largest eccentricity is thus 0.0074 in. The bending moments introduced by such deformations are negligibly small. The moments produced by the support conditions, however, are considerable. The membrane stress condition at the support ring is thoroughly disturbed resulting in large bending stresses in the direction of the meridians. It was specified that the outside surface of the shell should be a perfect sphere. For this reason the thickness of the shell increases inward and the middle surface, which bisects the thickness of the shell, deviates from a true spherical surface. The radii of curvature of this surface do not differ significantly from that of a sphere, except in the vicinity of the support ring where the change of its curvature is considerable and unfavorable. At points where the curvature of the middle surface changes suddenly, the change of the ring stresses is abrupt, and bending stresses are created in the direction of the meridians. By varying the shell thickness so that the middle surface is represented by a practically continuous function the abrupt changes in the ring stresses and the introduction of moments can be eliminated.

The dead and live loads were assumed to be symmetrical about the vertical axis of the sphere. The symmetry of the dead load and the smallness of the live load (about $\frac{1}{30}$ of the dead load) justified this assumption. The snow load was assumed to be 25 lb per sq ft at the top of the sphere, diminishing as the cosine of ϕ to zero at the equator. This assumption represents a total snow load about 60% larger than that used by the authors for the steel design, although that assumed by the authors was ample for any condition that would have occurred. The effect of this load, which is about $\frac{1}{20}$ of the dead load, is

[6] "Spannungen in Kugelschalen," by H. Reissner, *Müller Breslau Festschrift*, Kröner, Leipzig, 1912, p. 181.

small in the lower part of the shell. The computation of the wind stresses due to the given pressure distribution could be only approximate. These stresses, as compared to the dead-load stresses, have significant values only at the upper part of the sphere where the shell thickness, determined by construction considerations, assures small unit stresses. The advantageous relation between the dead-load and wind-load stresses assures that the shell deformations produced by the unsymmetrical wind loads are small enough to make the bending moments introduced by these deformations negligible.

The maximum membrane stresses in the concrete produced by the combined dead, live, snow, and wind loads do not exceed 188 lb per sq in. tension and 224 lb per sq in. compression.

At $\phi = 155°$ the shell is joined to the support ring which rests on eight columns. This ring is compressed radially by the horizontal component of the meridian stresses. The abrupt change of the ring stresses at the support ring produces bending and shearing stresses perpendicular to the middle surface of the shell. Temperature difference between the exposed shell and the inclosed support ring also has the same effect. Fig. 18(*g*), (*h*), and (*i*), shows the effect of the support ring on the shell. These stresses are due to dead, live, snow, and wind loads and 20° F temperature difference between the shell and the support ring. They were computed with consideration of the variation of the thickness of the shell.[7] Both vertical and horizontal components of the meridian stresses create bending stresses that are carried by the support ring in cooperation with the shell. The column reactions also effect additional stresses in the shell. The significant values of all of these stresses are localized in the vicinity of the support ring. They were determined by the application of theories developed primarily by Fr. Dischinger[8] and W. Flügge.[9] The maximum concrete unit stress produced by the combined effect of all loads and 20° F temperature difference between the shell and support ring is 574 lb per sq in. compression. To aid the distribution of the concentrated loads and stiffen the shell near the support, meridional ribs were arranged below $\phi = 140°$. Between the columns a shell 2.5 in. thick was suspended from the support ring. It has no statical purpose; it only completes the sphere.

The quantities of the structure are as follows:

Item	Concrete, in cu yd	Reinforcing steel, in lb
Shell	1,860	650,000
Ribs, framing and support ring	1,035	209,000
Suspended shell	29	6,000
Columns	56	25,000
Total	2,980	890,000

The writer believes that the costs given by the authors is high for Design No. 5. The construction of the lower part, up to $\phi = 120°$, where 70% of the

[7] "Über die Festigkeit Achsensymmetrischer Schalen," by J. Geckeler, *Forschungsarbeiten*, Heft 276, 1926, p. 19.

[8] "Die Rotationsschalen mit unsymmetrischer Form und Belastung," by Fr. Dischinger, *Der Bauingenieur*, Vol. 16, 1935, p. 393.

[9] "Die Statik und Dynamik der Schalen," by W. Flügge, Springer, Berlin, 1934, p. 43.

quantities are located, does not differ from common reinforced concrete jobs, except that the falsework supporting the forms of this part should be kept in place until the concrete of the entire structure hardens. The building of the upper part does not involve considerably greater difficulties than the construction of the numerous large shells which have overcome, successfully, the competition of framed steel structures. The writer is inclined to believe that bids based on the working drawings of the concrete design would have more than justified the cost of extensive model testing.

The authors showed ingenuity in the analysis and great structural skill in the layout and design of the steel structure. This is demonstrated by the fact that this unusual structure could be both fabricated and erected without difficulties.

Correction for *Transactions:* September, 1940, *Proceedings*, page 1284, caption for Fig. 2, change "7.94 ft" to "9.52 in."

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

AN INVESTIGATION OF STEEL RIGID FRAMES

Discussion

BY W. J. ENEY, ASSOC. M. AM. SOC. C. E.

W. J. ENEY,[13] Assoc. M. Am. Soc. C. E. (by letter).[13a]—In 1938 the writer completed a study of a rigid frame celluloid model to determine the effect of the shape of the knee and footing on the reactions, moments, and deflections. The data obtained are presented herein with the belief that they will prove useful in appraising the investigation reported by Messrs. Lyse and Black. Unfortunately the study was well under way before the American Institute of Steel Construction (A.I.S.C.) tests were started; otherwise the same prototype could have been used. The prototype (shown in Fig. 22) was one of the steel frames in a 96.5-ft span highway bridge erected over the Pennsylvania Railroad on City Line, Philadelphia, Pa. The general dimensions of the frame are given in Table 4 and Fig. 23. The model was constructed of sheet celluloid 0.125 in. thick to a geometric scale of 1 in. = 30 in. At corresponding sections, the width of the model, in inches, was made numerically equal to the cube root of the moment of inertia of the structure in foot units. The initial square, filleted, knee model (designated as Model L 2) was progressively altered at the knee and footing to obtain successively Models L 3, L 4, L 5, and L 6. The general nature of each of these changes is given in Fig. 24, together with the detailed dimensions. Influence lines for horizontal reaction, reaction moment, moment at and deflection of load point 10 L, were determined for both the hinged-base and fixed-base condition.

The deformeter apparatus used in the study is shown in Fig. 25 attached to Model L 3. The displacements of the load points were measured with a scale oriented in the direction of loads acting on the prototype. This scale, graduated to one hundred divisions per inch, was read easily with a watchmaker's loupe to an accuracy of plus or minus 0.002 in. The scale slides along a reference bar permitting the measurement of the displacement of all load points at one setting of a gage.

NOTE.—This paper by Inge Lyse, M. Am. Soc. C. E., and W. E. Black, Jun. Am. Soc. C. E., was published in November, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1941, by C. J. Posey, Assoc. M. Am. Soc. C. E.

[13] Asst. Prof., Civ. Eng., Lehigh Univ., Bethlehem, Pa.

[13a] Received by the Secretary December 26, 1940.

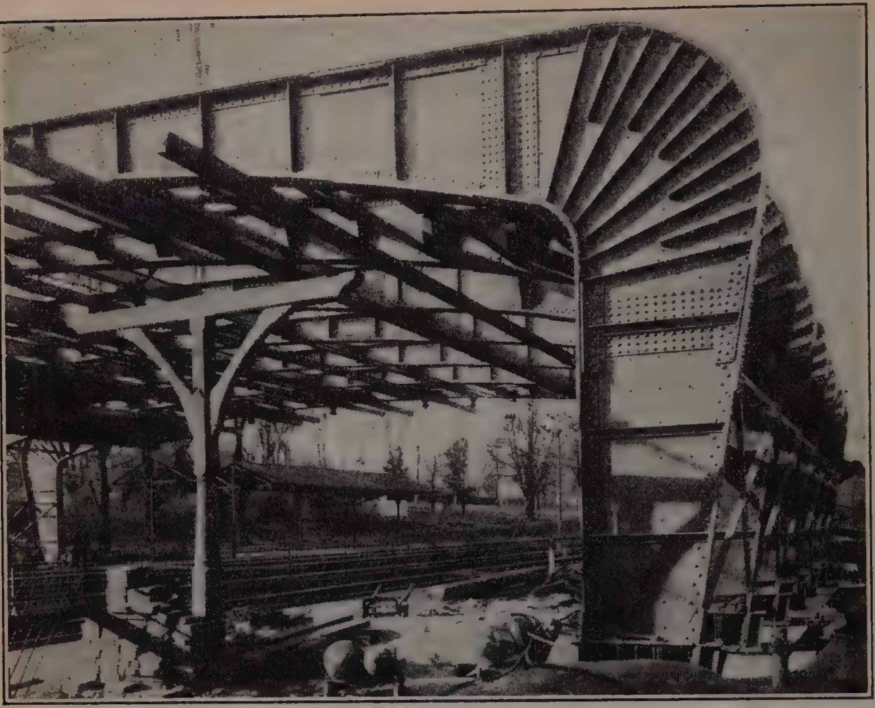


FIG. 22.—PROTOTYPE OF CELLULOID MODELS

Two types—a reaction gage and an internal deformeter—were used. They permitted the introduction of known shear, thrust, or rotation displacements of varying amounts. A shear displacement of 1.00 in. was introduced at the reaction, and a thrust displacement of 0.500 in. and rotation of 0.200 radians introduced at load point 10 L.

TABLE 4.—FRAME DIMENSIONS: WEB PLATES, ½ IN.; FOUR ANGLES, 6 IN. BY 6 IN. BY ¾ IN.; AND COVER PLATE, 14 IN. BY ¾ IN.

Axis division (see Fig. 23)	SECTION DETAILS		MOMENT OF INERTIA OF STRUCTURE, I_s		Width of model, I_s (in.)	COORDINATES[a] (FT)	
	No. of cover plates	Depth of section, back to back of angles (in.)	In.[4]	Ft[4]		X	Y
1	1	46.5	31,600	1.52	1.150	−0.30	3.70
2	1	59	53,400	2.57	1.370	−0.85	11.15
3	2	76	125,800	6.05	1.823	−0.35	18.20
4	2	56	65,200	3.14	1.465	6.10	20.00
5	2	42	35,600	1.72	1.200	12.50	20.85
6	2	33	21,350	1.03	1.010	19.00	21.40
7	2	27	14,000	0.675	0.877	25.50	21.80
8	2	23	10,100	0.486	0.786	32.00	22.10
9	2	20	7,500	0.361	0.712	38.50	22.40
10	2	18.5	6,450	0.311	0.673	45.00	22.50
Center line	..	....	....	....	....	48.25	22.40

[a] Coordinates for the division of center structure (see Fig. 23).

Computed Influence Ordinates.—The theoretical values presented for the hinged frame were computed by the Maxwell–Mohr method (Eq. 2, neglecting the shear term); and those for the fixed frame were computed by the Maxwell-

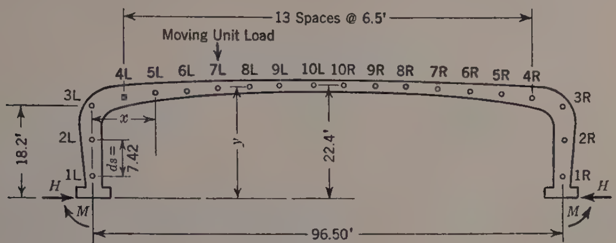


Fig. 23

Mohr method and checked by the method of Elastic Weights. The theoretical prototype had a round knee as in Model L 2, and a base corresponding to that of Model L 6. The frame was divided into section lengths varying from 6.30 ft to 7.42 ft for the algebraic summations in Eq. 2.

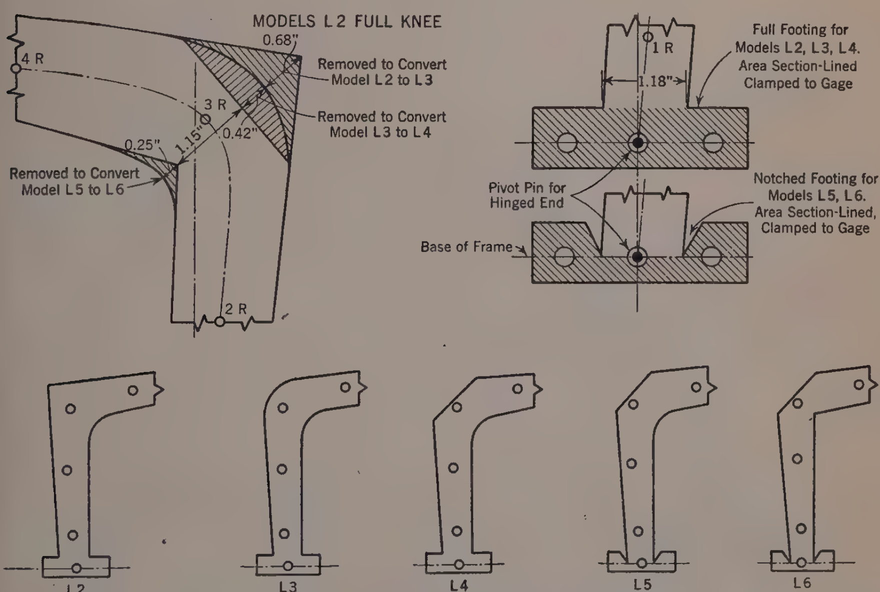


Fig. 24.—Details of Celluloid Models 12 to 16, Inclusive

Horizontal Reaction.—The influence lines for horizontal reaction are shown in Fig. 26. All of the observed values for the various modifications of the hinged end models lie along one curve. The ratio of the observed ordinates to the theoretical ordinates was 1.00 for the square-knee model L 2 and 0.975 for the round-knee model L 6. The corresponding ratios for the steel model were 0.985 and 0.965.

FIG. 25.—DEFORMETER APPARATUS AND MODEL L 3

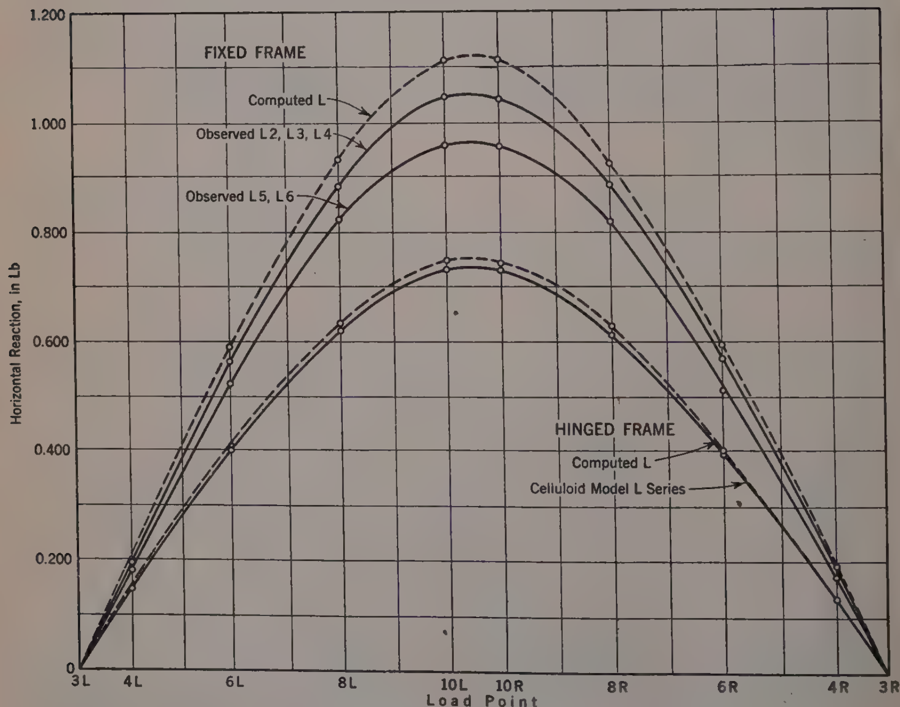


FIG. 26.—INFLUENCE LINES—HORIZONTAL REACTION, FIXED FRAME AND HINGED FRAME: CELLULOID MODELS

The effect of the modifications of the knee on the fixed frame horizontal reactions are negligible as the values for Models L 2, L 3, and L 4 are almost identical and one curve suffices for the three models. The observed ordinates are 92% of the theoretical ordinates. The reduction in the stiffness of the base

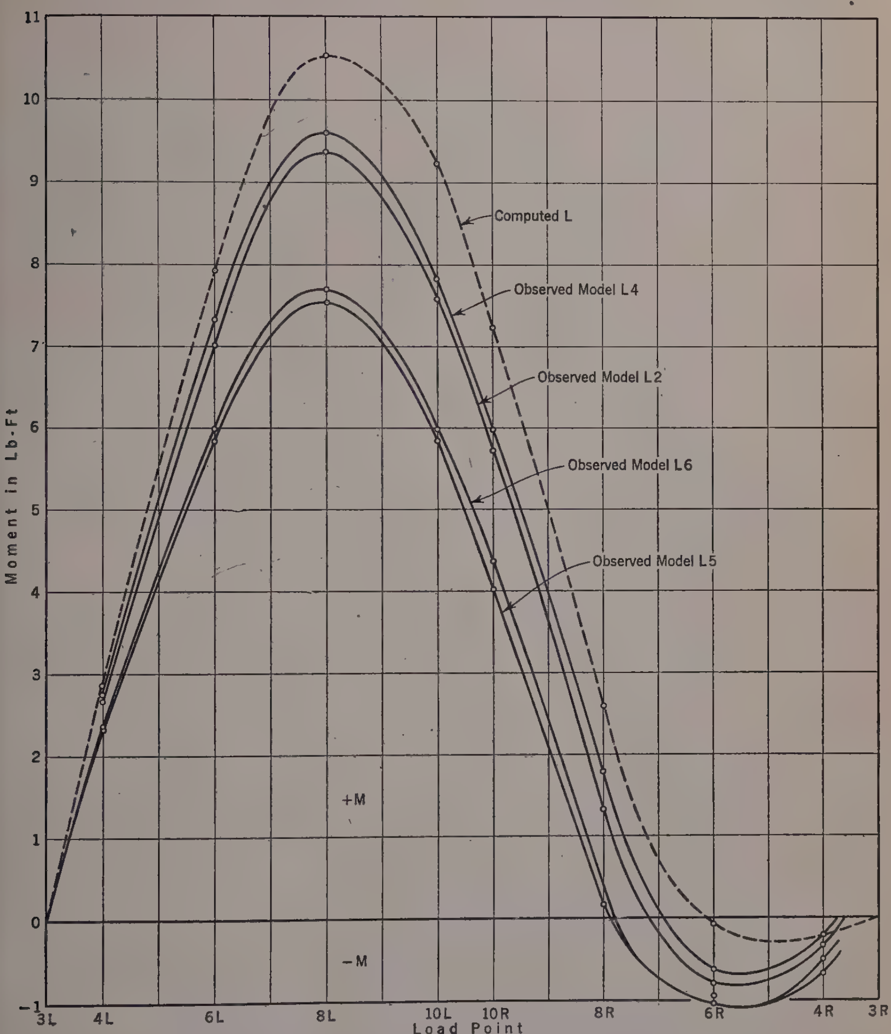


Fig. 27.—Influence Line, Moment at Right Footing, Fixed Frame

of the frame as represented by converting Model L 4 to Model L 5 caused an 8.5% additional reduction in the horizontal reactions.

Reaction Moments.—The influence lines for the moment at the right reaction are shown in Fig. 27. The computed maximum moment at the right support is approximately 11% greater than the observed value for Model L 3.

The modifications of the knee, changing Model L 2 to L 3 and then to L 4 and from L 5 to L 6, produced an increase in the moments of approximately 5.5%. Notching the footing, changing Model L 4 to L 5, caused a decrease of approximately 20% in the moments.

This 20% decrease in reaction moments is accompanied by a corresponding 8.5% decrease in the horizontal reaction, as stated. Since these produce

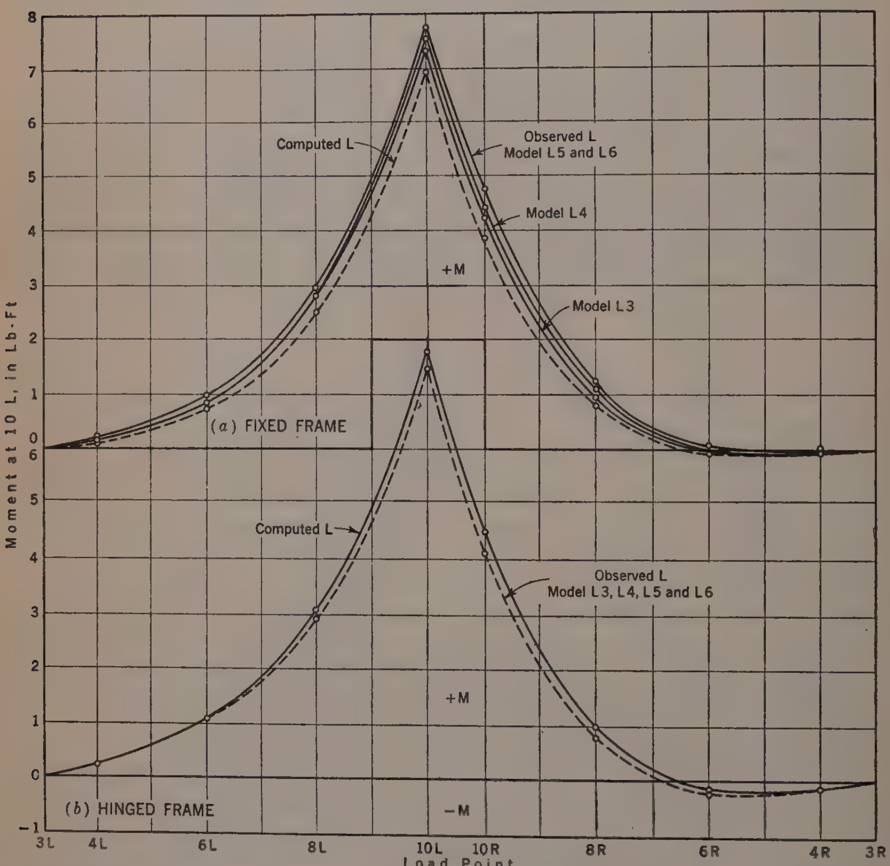


FIG. 28.—INFLUENCE LINES; MOMENT AT 10 L, FIXED AND HINGED FRAMES, CELLULOID MODELS; UNIT LOAD ON STRUCTURE

opposing moments in the frame, the reduction of the stiffness in the footing has little effect on moments at the knee or center of the frame. The moments at the knee, computed from the observed reactions, disagreed by approximately 5% with the theoretical moments.

Influence Lines for Thrust at Load Point 10 L.—The influence lines for thrust at load point 10 L were observed for Models L 3, L 4, L 5, and L 6 for both hinged and fixed frame conditions. The values were in agreement with the corresponding horizontal reactions and are therefore represented by Fig. 26.

Moment Near Center Frame.—The influence lines for moment at 10 L are given in Fig. 28. The effect of the modifications of the knee and the footing on the moment at 10 L in the case of the hinged end model was negligible—one curve fits the values for all models. In the case of the fixed frame the modification of Model L 2 progressively to L 6 caused an increase in the

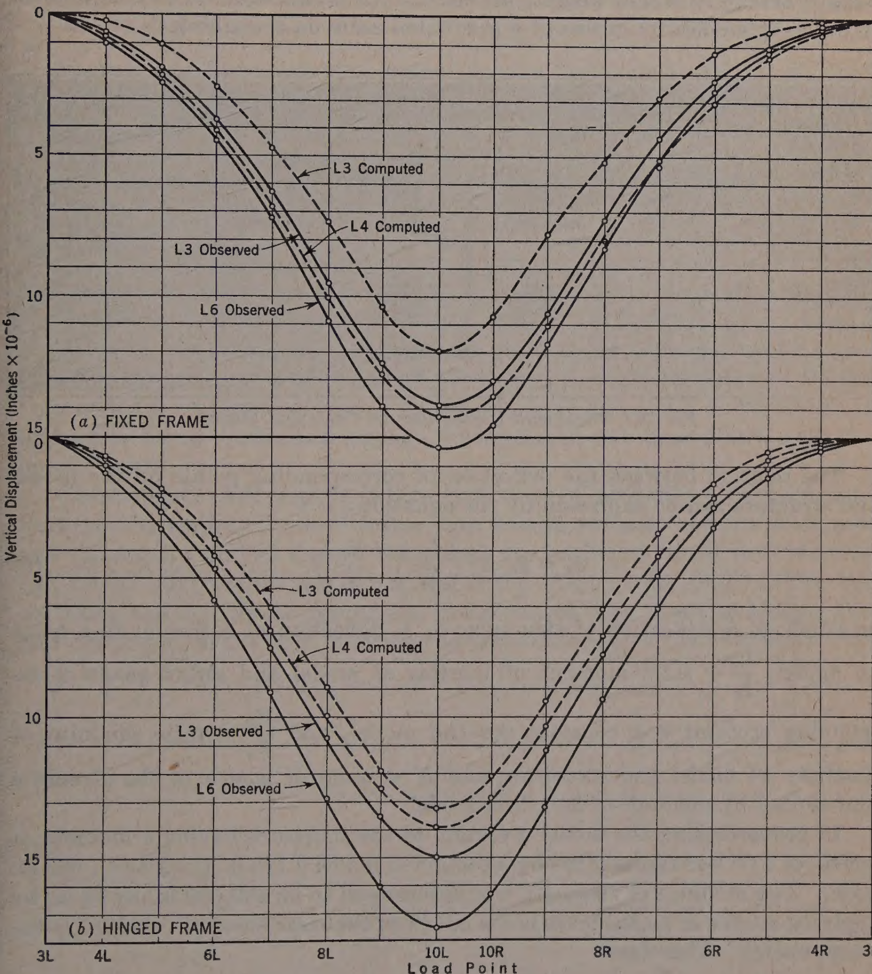


Fig. 29.—Influence Lines, Deflection of Point 10 L, Unit Load on Structure

moment at 10 L of approximately 5%. The comparison between the theoretical and observed ordinates reveals the same relationship as was found with the steel model—the observed moments were from 6% to 11% larger than the theoretical moments.

Influence Lines for Deflection.—Influence lines for the deflection of point 10 L for both the hinged and fixed end frame, with modified knees, are shown

in Fig. 29. These influence lines were obtained by applying a force to the celluloid frame model, with an elastic celluloid spring, and observing the deflections as shown in Fig. 30. The spring serves a dual purpose. First, it practically eliminates the annoying creep of the plastic model; and second, it serves as a convenient means of applying an accurate force to the model, eliminating the pulleys necessary where weights are used. An identical force can be applied to different models by means of a graduated scale on the spring.

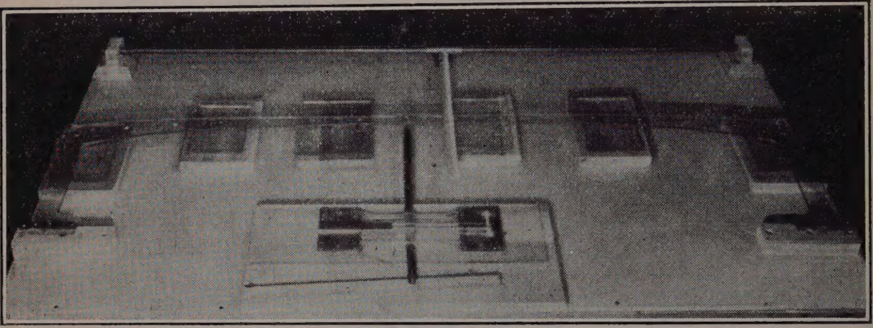


FIG. 30.—MEASURING DEFLECTIONS OF CELLULOID MODEL

The relation between the deflection of corresponding points on the model and structure can be expressed by the equation:

$$\delta_s = \delta_m N^3 \left(\frac{I_m}{I_s}\right)\left(\frac{E_m}{E_s}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which: δ_s = deflection of structure; δ_m = deflection of corresponding point on model; $\frac{I_m}{I_s}$ = ratio moment of inertias of model and structure at corresponding sections = a constant for the model, and $\frac{E_m}{E_s}$ = ratio modulus of elasticity of model and structure; and N = units of length in the structure represented by one unit of length in model.

In proportioning the model a section of the structure having a moment of inertia of 1 ft^4 was represented by a section of model 0.125 in. thick and 1.000 in. wide. The modulus of elasticity was determined to be 380,000 lb per sq in. by applying a force to a cantilever beam cut from the same sheet of celluloid as the frame model. Therefore:

$$\delta_s = \delta_m N^3 \left(\frac{I_m}{I_s}\right)\left(\frac{E_m}{E_s}\right) = \delta_m 30^3 \left(\frac{0.125 \times 1^3 \times \frac{1}{12}}{1 \times 12^4}\right)\left(\frac{380{,}000}{30 \times 10^6}\right) = \frac{\delta_m}{5{,}820}.$$

The frame model, therefore, deflects 5,830 times the deflection of the structure for equal loads.

The data presented in Fig. 29 show that the shape of the knee has a small effect on the frame deflections. Chamfering the outside of the knee and re-

moving the fillet, converting Model L 3 to L 6, increased the hinged frame deflection at point 10 L approximately 15% and the fixed frame deflections approximately 10%.

The computed influence lines marked "L 3 computed" were determined by using the method of dummy unit loading and theoretical moments computed by the elastic theory, whereas the one marked "L 4 computed" was determined by the tangent offset method in which a conjugate beam was loaded with the $\frac{M}{I}$-diagram and the moments obtained by the mechanical analysis of Frame L 4. These computed deflections did not include the effect of shear or rib shortening. The deflection would be increased 0.6% due to rib shortening and approximately 2.0% due to shear. The observed deflections are from 6% to 17% greater than the computed deflections. Messrs. Lyse and Black found the observed deflections for the hinged, round knee steel model to be 11% greater than the computed deflections.

Comparing the observed deflection of Model L 3 one finds the fixed frame deflects about 90% as much as the hinged frame. This slight increase in rigidity is easily destroyed by a very slight rotation of the base. The largest observed rotation of the footing of the hinged model amounted to 0.00206 radians for a unit load of 10 L. As Messrs. Lyse and Black show, unless the footing is firmly restrained against rotation the flat base frame acts as a hinged frame.

Conclusions.—Although it is true that there is probably a vast difference between the action of a steel frame with a knee stiffened radially in Fig. 22, and the round celluloid knee of the model, the results obtained with the celluloid model nevertheless are indicative of the effect of the shape of the knee on frame action. These results are in agreement with those of Messrs. Lyse and Black and, in addition, indicate the probable effect of the knee and footing on the fixed frame.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

TRANSATLANTIC SEAPLANE BASE, BALTIMORE, MARYLAND

Discussion

By Karl Terzaghi, M. Am. Soc. C. E.

Karl Terzaghi,[4] M. Am. Soc. C. E. (by letter).[4a]—During the construction of the seaplane base at Baltimore, the author had an unusual opportunity to observe the gradual consolidation of an hydraulic mud fill. Under the heading "The Mud" he mentions that he used wicks for the purpose of accelerating the drainage of the mud fill. A brief statement of the mode of application of the wick principle, and of the results obtained by means of this method, would certainly be of general interest. The hangar floor consists of a bituminous concrete placed on a rolled sand fill. The sand fill seems to be supported by a granular fill which in turn rests on an incompletely consolidated mud fill. The writer would appreciate some information on the rate at which the floor settles and on the method which is used for the periodic readjustment of its position, provided such readjustment is required.

Corrections for *Transactions:* October, 1940, *Proceedings*, p. 1488, delete line 34.

Note.—This paper by W. Watters Pagon, M. Am. Soc. C. E., was published in October, 1940, *Proceedings*.

[4] Dr. Ing., Lecturer, Harvard Univ., Cambridge, Mass.

[4a] Received by the Secretary January 27, 1941.